Handbook of International Research in Mathematics Education

Second Edition

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Routledge
Taylor & Francis Group
NEW YORK AND LONDON

First edition published by Lawrence Erlbaum Associates, 2002
This edition first published 2008
by Routledge
270 Madison Ave, New York, NY 10016

Simultaneously published in the UK
by Routledge
2 Park Square, Milton Park, Abingdon, Oxon OX14 4RN

Routledge is an imprint of the Taylor & Francis Group, an informa business

Typeset in Galliard by EvS Communication Networx, Inc.
Printed and bound in the United States of America on acid-free paper by Sheridan Books, Inc.

Library of Congress Cataloging in Publication Data
Handbook of international research in mathematics education / edited by Lyn D. English ; associate editors, Maria Bartolini Bussi ... [et al.]. — 2nd ed.
p. cm.
1. Mathematics—Study and teaching—Research. I. English, Lyn D. II. Bartolini Bussi, Maria G. (Maria Giuseppina)
QA11.2.H36 2008
510.71—dc22
2007041882

ISBN 10: 0-8058-5875-X (hbk)
ISBN 10: 0-8058-5876-8 (pbk)
ISBN 10: 0-203-93023-1 (ebk)

ISBN 13: 978-0-8058-5875-4 (hbk)
ISBN 13: 978-0-8058-5876-1 (pbk)
ISBN 13: 978-0-203-93023-6 (ebk)

Contents

28 Semiotic mediation in the mathematics classroom

Artifacts and signs after a Vygotskian perspective

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ARTIFACTS AND COGNITION

The construction and the use of artifacts[1]—in particular complex artifacts—seems to be characteristic of human activities, but even more characteristic of human beings seems to be the possibility of the contribution of such artifacts beyond the practical level, e.g., their contribution at the cognitive level. In the field of practice, tools have always played a crucial role; often practical problems are related to the use of an artifact, so that often the solution process of a given problem and the design of an artifact, expressly designed to support that solution, have been mutually developed. At best extent theoretical knowledge may be considered as originating from such a mutual shaping in a long-term process, the traces of which can sometimes be reconstructed.

Norman's book, *Things That Make Us Smart* (1993) exactly hints at the double nature of what he calls *cognitive artifacts*:

- the *pragmatic* or *experiential* feature (i.e., *outward* orientation that allows modification of the environment);
- the *reflexive* feature (i.e., *inward o*rientation that makes users smart).

This double nature and double orientation will run as a leitmotiv through this entire chapter.

The idea of artifact is very general and encompasses several kinds of objects produced by human beings through the ages: sounds and gestures; utensils and implements; oral and written forms of natural language; texts and books; musical instruments; scientific instruments; tools of the information and communication technologies. The contribution of artifacts to education is not novel; books are the main artifacts used in schools, but let's not forget paper and pencil and the blackboard. More generally, the passage from the sphere of practice to that of intellect and vice versa, may be considered one of the basic motor of evolution and progress.

Certainly the language in all its forms, oral and written, has a central place among the artifacts produced and elaborated by human beings. The insightful studies concerning the development from oral to written culture tell us the story of a fascinating evolution of ways of thinking. As the seminal studies of McLuhan (1962) and later of Ong (1967/1970) have shown, the emergence of writings not only strengthened mind's capacities but also it can be considered the source of the evolution of specific thought schema. In fact, the introduction of writing modified the traditional schemes of communication through the introduction of a new communication means. Oral communication is pragmatic in the sense that it aims to make the interlocutors share a common experience as shown by the extensive use of deictic forms. One of the main consequences, thus, according to Ong (1967/1970) is that it may present difficulties in elaborating abstract concepts detached from concrete and well recognizable situations. The introduction and the development of the written form of language seem to be a source of logic-deductive schemas per se, without a precise reference to a real situation (Goody, 1987/1989).

The passage from the oral to the written form of language was the origin of a great change; at a first glance writing can be simply considered a way to implement oral expression, with the advantage that what is said may be recorded. Once written, an utterance can be read, and in so doing, it can be "said" again and again, whenever and by whomever. But to consider writing just a simulation of oral expression would be limited and misleading. The history of writing from the first records on clay tablets onward shows evidence of its contribution in transforming thinking:[2] "... writing creates the difference: not only in thinking expression, but also and over all in how such thinking is thought. (authors' translation)" (Goody, 1987/1989 p. 266). This is consistent with the classic reference to Luria (1976, p. 161): "our investigation, [...] showed that as the basic forms of activity change, as literacy is mastered, and a new stage of social and historical practice is reached, major shift occur in human mental activity. These [...] involve the creation of new motives for action and radically affect the structure of cognitive processes."

As far as mathematics is concerned, it is worthwhile to remember that the use of writing may be, as some historians suggest (Cambiano, 1997), related to the naissance of deductive reasoning in the geometry field, with Thales as one of its beginners and Euclid the master.

The conclusion that can be drawn from the complex discussion about the implication of the development of writing and of its use concerns the role played by such an artifact in the emergence of what is called rational thinking that is thinking based on abstract ideas, general /universal statements and deductive reasoning. It is not surprising that similar distinctions can be found in recent studies concerning modern cultural contexts. In fact, the classic studies carried out by Nunes, Schlimann, and Carraher (1993) confirm and analyse the difference between the written and the oral, highlighting how it is possible to relate to this distinction the deep differences between informal and formal mathematics, i.e., between what they call "streets and school mathematics."

The example of writing and its history of shaping ways of thinking, and mathematics in particular, leads one to reflect on the cognitive role of representations and to the basic remark that any representation comes to life because of a human construction that makes it possible, in other words any representation is supported by an artifact. Human beings have produced many such artifacts supporting representations of different kind. A particular case is that of scientific instruments among which the compass is probably one of the best known.

The strict link between the use of instruments like a ruler and a compass is easily recognizable in the origin of classic Euclidean geometry. The intimate relationship between the functioning of the brain and the body experience (without or with instruments) even in the most sophisticated and abstract evolution of mathematics is now commonly recognized (Arzarello, 2006). The gesture of tracing can be considered at the origin of the idea of line (both a straight line and a circle), but mainly what seems to us more interesting is the fact that this gesture is to be related to the use of a particular artifact: for instance, a rope either stretched between two nails or turned around a pivot, or a rule or a compass. In spite of its very source, however, the process of construction of mathematical knowledge is not directly and simply related to practice, neither is it directly related to the use of artifacts.

Perhaps one of the most evident examples is that of the circle. The definition of circle—the geometric figure—is certainly related to the use of compass, but the passage from the use of compass to trace round shapes to the conception of the circle—consistent with the geometrical definition—as "the locus of the points equidistant from the centre" is not immediate (Bartolini Bussi, Boni M., & Ferri, 2007).

RABARDEL'S INSTRUMENTAL APPROACH

The previous examples show that the relationship between artifacts and knowledge is complex and asks a careful analysis in order to avoid useless oversimplification. In the last decades, a new kind of artifact has become readily available: the tools of the information and communication technologies. It is trivial to say that these tools have empowered and changed human way of thinking. Their use in schools has, on the one hand, encouraged educators to reconsider curricula and, on the other hand, called attention to the relationships between students and computers. This may explain the diffusion of educational studies framed by instrumental approaches (Rabardel, 1995) whereby the complexity of the context of students' activity is investigated. Rabardel's instrumental approach is based on the distinction between *artifact*[3] and *instrument*. Such a distinction leads one to analyze separately the potentialities of an artifact and the actual use that is realized through it, hence not only to separate what is related to the intention of the designer and what actually occurs in practice but also to emphasize the distinction between objective and subjective perspectives. According to Rabardel's terminology, *artifact* is the material or symbolic object per se; sometimes only a part of a complex artifact may be in focus that is designed according a particular goal, and, for this reason, embedding a specific knowledge. The *instrument* is defined by Rabardel as a mixed entity "made up of both artifact-type components and schematic components that we call *utilization schemes*. This mixed entity is born of both the subject and the object. It is this entity which constitutes the instrument which has a functional value for the subject" (Rabardel & Samurçay, 2001). The utilization schemes are progressively elaborated in using the artifact in relation to accomplishing a particular task; thus the instrument is a construction of an individual, it has a psychological character and it is strictly related to the context within which it originates and its development occurs. The elaboration and evolution of the instruments is a long and complex process that Rabardel names *instrumental genesis (genèse instrumentale,* Rabardel, 1995, p.135 ff.). Instrumental genesis can be articulated into two processes:

- *Instrumentalisation*, concerning the emergence and the evolution of the different components of the artifact, e.g. the progressive recognition of its potentialities and constraints.
- *Instrumentation*, concerning the emergence and development of the utilization schemes.

The two processes are outward and inward oriented, respectively from the subject to the artifact and vice versa, and constitute the two inseparable parts of instrumental genesis. The utilization schemes may or may not be consistent with the pragmatic goals for which the artifact has been designed, basically they are related to the phenomenological experience of the user, and according to this experience they may be modified or integrated. Rabardel theorizes the impact of the use of tools on cognitive activity: the use of a tool is never neutral (Rabardel & Samurçay, 2001), on the contrary it originates a re-organization and mobilization of cognitive structures, as shown in the classic example of the evolution in the conceptualization of space during the activity mediated by a robot. The social dimension is addressed by Rabardel in describing the interplay between individual schemes of utilization and social schemes, as elaborated and shared in communities of practice. In particular, explicit training processes may give rise to an appropriation by subjects.

Rabardel's approach has been developed within the field of cognitive ergonomy, hence does not aim at coping all the needs of school education research. It has, however, become popular and has been employed in different research studies concerning mathematics education and in particular the pedagogy of use of CAS environments. Such an approach showed very powerful, shading light on some crucial aspects mainly related to the possible discrepancies between pupils' behaviours and teachers' expectations (Lagrange, 1999; Artigue, 2002; Guin, Ruthven & Trouche, 2005; Drijvers & Trouche, in press). As explained in the following, the instrumental approach has to be further elaborated in order to match the complexity of classroom activity and in particular that of the teaching learning of mathematics, in fact it may provide a frame to analyse the cognitive processes related to the use of a specific artifact and consequently what will be considered its semiotic potential.

VYGOTSKY'S APPROACH TO ARTIFACTS

The notion of cognitive artifacts, introduced by Norman (1993), as well some of the related ideas have sound historical basis in the socio-historic school, in particular in the work of Vygotsky (and his successors: Luria, 1976; Leont'ev, 1976/1964, see also Wertsch, 1985; Wertsch & Addison Stone, 1995). A Vygotskian perspective including a developmental dimension interprets the functioning of cognitive artifacts as a main element of the learning and for this reason seems to offer an adequate framework to study the use of artifacts in the field of education. In the following, we shall limit ourselves to sum up briefly some elements, in order to go very fast to the specific aim of this chapter, i.e., a precise definition of *tool of semiotic mediation* and its application to research studies in the mathematics classroom.

Vygotsky, contrasting humans and animals, postulated two "lines" for the genesis of human mental activity: the natural line (for elementary mental functions) and the social or cultural line (for the higher mental functions). The specific nature of human cognitive development is the product of the "interweaving of these two lines." What seems to be interesting especially when we study its development during the school age, and in particular within the school context, is the evolution of human cognition as effect of social and cultural interaction. These two elements (social and cultural) found a counterpart in two key notions introduced by Vygotsky: that of *zone of proximal development* and that of *internalization*. These notions play a crucial role in the use of artifacts that Vygotsky postulates in relation to the internalization process.

THE ZONE OF PROXIMAL DEVELOPMENT: DEVELOPMENT AND LEARNING

The notion of the Zone of Proximal Development (ZPD) models the learning process through social interaction and it is defined by Vygotsky as "the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under the adult guidance or in collaboration with more capable peers" (1978, p. 86). As this definition clearly states, development is possible because of a collaboration between one individual, whose cognitive attitude presents a potential in relation to change, and another individual (or a collectivity) who intentionally cooperate to accomplish a task or to pursue a common aim. Without entering a debate concerning the relationship between learning and development, we assume that the asymmetry of the definition of the ZPD nicely fits in the school context with the intrinsic asymmetry of the relationship between pupils and teacher in relation to knowledge. Similarly, we assume that the notion of ZPD clearly highlights the need of harmonizing pupil's potential attitude to learn with the teaching action of the teacher. In ZPD, cognitive development is modelled by the process of internalization.

THE INTERNALIZATION

Internalization, as defined by Vygotsky as "the internal reconstruction of an external operation" (1978, p. 56), describes the process of construction of individual knowledge as generated by socially shared experiences. The relationship between internal (or psychic) and external processes (related to external social interaction) is a long-standing issue in psychology, where different models has been developed. In particular, Vygotsky's approach, subsequently developed by other authors (e.g., Leont'ev, 1976; Luria, 1976), assumes a strict dependence of internal from external processes, a genetic or developmental relationship, according to which external process are transformed to generate what Vygotsky (1981) called "higher mental functions." As Vygotsky expresses it:

> For the first time in psychology, we are facing the extremely important problem of the relationship of external and internal mental functions … everything internal in higher forms was external, i.e. for others it was what it now is for oneself. Any higher mental function necessarily goes through an external stage in its development because it is initially a social function. This is the centre of the whole problem of internal and external behaviour … When we speak of a process, "external" means "social". Any higher mental function was external because it was social at some point before becoming an internal, truly mental function. (Vygotsky, 1981, p. 162, cited by Wertsch & Addison, 1985, p. 166)

As assumed by a *fundamental Vygotskian hypothesis*, the internalization process has two main aspects: it is essentially *social* and it is directed by *semiotic processes.* In fact, as a consequence of its social nature, external process has a communication dimension involving production and interpretation of signs. That means that the internalization process has its base in the use of signs[4] (primarily natural language, but also any type of signs, from gestures to the most sophisticated such as the mathematical semiotic systems) in the *interpersonal space* (Cummins, 1996). For this reason the analysis of the internalization process may be centred on the analysis of the functioning of natural language and of other semiotic systems in social activities (Wertsch & Addison Stone, 1985, pp. 163–166).

SYSTEMS OF SIGNS IN THE PROCESS OF INTERNALIZATION

As is well known, Vygotsky focused on the study of the functioning of natural language, that is of the semiotic processes related to the apprentice and use of language (in particular the use of words, considered by Vygotsky the unit of analysis). In a Vygotskian perspective, the use of words and discourse forms may be interpreted according to the general assumption that a child's development is a progressive appropriation and reflexive use of ways of behaving that others used with respect to her.

The use of signs in accomplishing a task has a twofold cognitive function: the subject produces signs related directly to accomplish the task and to communicate with the diverse partners collaborating in the task. In this second case, the production of signs is strictly related to the process of interpretation that allows exchange of information and consequently communication.

Higher mental functions, or cognitive skills as Wertsch and Addison Stone call them (1985, p. 164), develop through the production and interpretation of signs: in particular, speaking (or writing) and interpreting what is said (or written), in other words socially interacting through communicating. "Thinking and making sense (in society as well as in schools) has to be conceived of as *sociosemiotic process* in which oral and written *texts* […] constantly interact in order to bring about improved texts on the part of the interlocutors or even merge into a revised text as a final product of the whole group" (Carpay & van Oers, 1999, p. 303). This remark is needed and crucial, because the cognitive function of using signs changes according to the function that signs have in the activity. In particular, taking into account the specificity of school activities, "the utterances of each of the interlocutors are determined by the position they occupy in a certain specific social formation" (Carpay & van Oers, 1999, p. 302) or their particular social position, as in the case of the asymmetrical position of teacher and students in respect to mathematics. This fundamental distinction will clearly emerge as we introduce the notion of *tool of semiotic mediation*.

ARTIFACTS AND SIGNS

Vygotsky pointed out that in the practical sphere human beings use artifacts,[5] reaching achievements that would otherwise have remained out of reach, while mental activities are supported and developed by means of signs that are the products of the internalization processes and that in Vygotskian terminology are called *psychological tools*. The former are directed outward, whilst the latter are oriented inward. This perspective is resonant with what is discussed above. The fundamental role of artifacts in cognitive development is widely recognized, but differently from other psychological approaches that clearly separate the technological and concrete artifacts from signs; the Vygotskian perspective claims an analogy between them.

> The invention and use of signs as auxiliary means of solving a given psychological problem (to remember, compare something, report, choose, and so on) is analogous to the invention and use of tools in one psychological respect. The sign acts as an instrument of psychological activity in a manner analogous to the role of a tool in labour. (Vygotsky, 1978, p. 52)

In most of the subsequent literature, signs have been interpreted as linguistic signs (Hasan, 2005) due to the greater importance attached by Vygotsky to language. Yet, without elaborating on it, Vygotsky himself suggested a list of possible examples, beyond language: "The following can serve as examples of psychological tools and their complex systems: language; various systems for counting; mnemonic techniques; algebraic symbol systems; works of art; writing; schemes, diagrams, maps, and mechanical drawings; all sorts of conventional signs; etc" (Vygotsky 1981, p. 137). Some issues of this list are related to mathematics and therefore relevant to the field of mathematics education. This fact is not surprising if we think of the particular nature of mathematics objects that require an external representation to be managed (Duval, 1995).

MEDIATION

As already said, analogy between signs and artifacts is based on the mediation function that both may have in accomplishing a task. Because of the centrality of this function in the following discussion, we think that few words are needed to clarify what is meant for mediation. As Hasan points out:

> the noun *mediation* is derived from the verb *mediate*, which refers to a process with a complex semantic structure involving the following participants and circumstances that are potentially relevant to this process: [1] someone who mediates, i.e. a *mediator*; [2] something that is mediated; i.e. a *content/force/energy* released by mediation; [3] someone/something subjected to mediation; i.e. the "*mediatee*" to whom/which mediation makes some difference; [4] the circumstances for mediation; viz,. (a) the means of mediation i.e. *modality*; (b) the location i.e. *site* in which mediation might occur.

> These complex semantic relations are not evident in every grammatical use of the verb, but submerged below the surface they are still around and can be brought to life through paradigmatic associations i.e. their systemic relations: we certainly have not understood the process unless we understand how these factors might influence its unfolding in actual time and space. (Hasan 2002)

Mediation is a very common term in educational literature. The term is used just referring to the potentiality of fostering the relation between pupils and mathematical knowledge, and mostly related to the accomplishment of a task. The idea of mediation in relation to technologies is widely present in the current mathematic education literature. Starting from the claim that is necessary to overcome the dichotomy between human beings and technologies, the unity between human and media becomes the basic goal: the tool becomes transparent (Meira, 1998), the violin is one with the violinist, as Moreno-Armella and Santos-Trigo note (chapter 14, this volume). Borba and Villarrael (2005) discuss the potentialities of these circumstances and introduced the term *humans-with-media*. Bottino and Chiappini (2002) emphasize the fact that artifacts not only enable but also constrain the subject's action in respect to the object. "The introduction of a new artifact in an activity influences both the norms regulating participant's interaction in the activity and the roles that participants can assume" (p. 762). In this way, they address some aspects of the complexity of interactions within the classroom, in particular peer interaction mediated by software. More than others, Noss and Hoyles (1996, p. 6) emphasize the perspective of communication: the mediation function of the computer is related to the possibility of creating a communication channel between the teacher and the pupil based on a shared language.

All these positions are consistent with the Hasan's model, although not all the elements receive the same attention. Hasan's model is explicitly put in a Vygotskian frame and, as we shall explain in the following, encompasses all the relevant elements for modelling teaching learning activities from a semiotic point of view. In order to do that, further elaboration of Vygotskian ideas, in particular concerning the nature and the role of the mediator and the characteristics of circumstances, is needed.

A PARTICULAR TYPE OF MEDIATION: SEMIOTIC MEDIATION

According to the fundamental Vygotskian hypothesis stated above, within the social use of artifacts in the accomplishment of a task (that involves both the mediator and the mediatees) shared signs are generated. On the one hand, these signs are related to the accomplishment of the task, in particular related to the artifact used, and, on the other hand, they may be related to the content that is to be mediated ([2] in Hasan's model). Hence, the link between artifacts and signs overcomes the pure analogy in their functioning in mediating human action. It rests on the truly recognizable relationship between particular artifacts and particular signs (or system of signs) directly originated by them, as the example of a move from abacus to the positional representation of a number will illustrate below.

The link between artifacts and signs is easily recognized, yet what needs to be emphasized and better explained is the link between signs and the content to be mediated and the way in which these links can be exploited in an educational perspective.

A CULTURAL ARTIFACT AS A TOOL OF SEMIOTIC MEDIATION

The relationship between artifacts and signs within the solution of a task has a counterpart in the historic /cultural development of knowledge, where such a relationship is crystallized in the shared knowledge of a society (Leont'ev, 1976/1964, p. 245) and expressed by the shared systems of signs, either the natural language or the specialized system of signs of different scientific domains. A potential link with artifacts is, in principle, possible to be reconstructed even if completely lost (Wartofsky, 1979). Our approach elaborates on this assumption in an educational perspective and in particular within the school setting. The main point is that of exploiting the system of relationships among artifact, task and mathematical knowledge. On the one hand, an artifact is related to a specific task (see the above definition of Rabardel's notion of instrument) that seeks to provide a suitable solution. On the other hand, the same artifact is related to a specific mathematical knowledge. In this respect, a double semiotic link is recognizable between an artifact and both a task and a piece of knowledge. In this sense one can talk of polysemy of an artifact. In principle, the expert can master such a polysemy, and, most of the time, this may happen unconsciously.

Polysemy of the artifact and emergence of signs

The polysemy of the artifact finds a counterpart in the coexistence of different systems of signs, sometimes parallel or partially overlapping, often merging into a comprehensive system, according to the evolutionary model of Wartofsky.[6]

On the one hand, the relationship between artifact and knowledge may be expressed by signs, culturally determined, produced by cultural development, and crystallizing the meaning of the operations carried out with the artifact.[7]

On the other hand, the relationship between the artifact and the task may be expressed by signs,[8] often contingent to the situation determined by the solution of the particular task. In any case, a main characteristic of these signs is that their meaning maintains a strong link with the operations accomplished. Gestures, drawings, or words may be the different semiotic means used to produce these signs, the production of which may be spontaneous or explicitly required by the task itself. It may also happen that the expert provides the novices with these signs. This last case seems relevant from an educational perspective (Douek, 1999).

The relationship (see Figure 28.1) between these two parallel systems of signs, hinged on the artifact, is certainly neither evident nor spontaneous. For this very reason we state the following assumption:

> the construction of this relationship becomes a crucial educational aim that can be realized promoting the evolution of signs expressing the relationship between the artifact and tasks into signs expressing the relationship between artifact and knowledge.

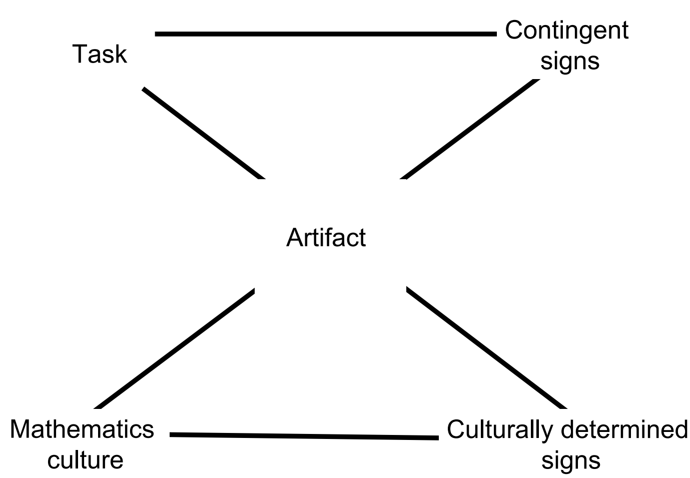

Figure 28.1 Polysemy of an artifact.

Signs sprouting from activities with the artifact are socially elaborated: in particular, they can be intentionally used by the teacher to exploit semiotic processes, aiming at guiding the evolution of meanings within the class community. In particular, the teacher may guide the evolution towards what is recognizable as mathematics. In our view, that corresponds to the process of relating personal senses[9] (Leont'ev, 1964/1976, p. 244 ff.) and mathematical meanings, or of relating spontaneous concepts and scientific concepts (Vygotsky, 1934/1990, p. 286 ff.).

In so doing, the teacher will act both at the cognitive and the metacognitive level, both fostering the evolution of meanings and guiding pupils to be aware of their mathematical status (see also mathematical norms,Cobb, Wood, & Yackel, 1993).

In summary, on the one hand, personal meanings are related to the use of the artifact, in particular in relation to the aim of accomplishing the task; on the other hand, mathematical meanings may be related to the artifact and its use. This double semiotic relationship will be named *the semiotic potential of an artifact.*

Because of this double relationship, the artifact may function as a semiotic mediator and not simply as a mediator, but such a function of semiotic mediation is not automatically activated; we assume that such a semiotic mediation function of an artifact can be exploited by the expert (in particular the teacher) who has the awareness of the semiotic potential of the artifact both in terms of mathematical meanings and in terms of personal meanings. Such evolution is fostered by the teacher's action, guiding the process of production and evolution of signs centred on the use of an artifact. In terms of mediation, we can express this complex process as follows: the teacher acts as mediator using the artifact to mediate mathematical content to the students. In our view: *the teacher uses the artifact as a tool of semiotic mediation*

Because of the cultural significance of this process, we call the teacher a cultural mediator. We do not refer only to the concrete act of using a tool to accomplish a task, rather to the fact that new meanings, related to the actual use of a tool, may be generated, and evolve, under the guidance of the expert (i.e., the teacher).

Thus any artifact will be referred to as *tool of semiotic mediation* as long as it is (or it is conceived to be) intentionally used by the teacher to mediate a mathematical content through a designed didactical intervention. In fact, the use of the artifact has to be fully integrated in the classroom activities, i.e., the use of the artifact has to be "orchestrated"[10] as Trouche expresses it (2005, p. 123). The key point according to our hypothesis is that the twofold role played by the artifact, both as a means in accomplishing a task, and as a tool of semiotic mediation to accomplish a didactical objective, can be fully exploited. The role of the teacher is then crucial and not incidental and the teaching sequence has to present certain peculiarities. The following section is devoted to describing the main characteristics of a teaching sequence as they are implied by the previous assumptions; we call it *didactical cycle*. According to the general methodology of our research, it can be considered as result coming form the complex process where the design of a teaching experiment and the reflection on its results do not follow a linear order, rather nurture each other "theory and practice are generated together" (Arzarello, Bartolini Bussi 1998, p. 249). For this very reason, the following didactic cycle provides a frame both for designing and for analysing a teaching sequences.

DIDACTICAL CYCLE

The structure of a teaching sequence may be outlined as an iteration[11] of a cycle where different typology of activities took place, aimed to develop different components of the complex semiotic process described above:

Activities with artifacts: students are faced with tasks to be carried out with the artifact. Generally used to start a cycle, this type of activities promote the emergence of specific signs in relation to the use particular artifacts/tools; in fact working in pairs, or small groups, promotes social exchange, accompanied by words, sketches, gestures, and the like.

Individual production of signs (e.g., drawing, writing, and the like). Students are individually engaged in different semiotic activities, mainly concerning written productions. For instance, after using the artifact, students are asked to write at home individual reports on their own experience and reflections, including doubts and questions arisen. In the case of young children, specific tasks are designed asking to draw (for instance explain through a drawing the functioning of an artifact). They may be asked to write, on their own math notebook, the final shared mathematical formulation of the main conclusions coming from the collective discussion (see below). All these activities[12] are centred on semiotic processes, i.e., the production and elaboration of signs, related to the previous activities with artifacts. Although, social interchange during activities with the artifact, or the following collective discussions, also involve semiotic processes, this type of activity differs in that it requires a personal contribution in order to produce written texts and consequently graphic signs, which for their very nature start to be detached from the contingency of the situated action. Because of their nature and differently to other signs, like gestures, written signs (in particular words) are permanent and can be shared, may be involved in collective discussions, and even become objects of discussion. This may make them evolve.

Collective production of signs (e.g., narratives, mimics, collective production of texts and drawings). Among others, collective discussions play a crucial role, specifically the particular type of collective discussion (see *Mathematical Discussion,* Bartolini Bussi, 1998). Collective discussions play an essential part in the teaching and learning process where the core of the semiotic process, on which teaching/learning is based, will take place. In a mathematical discussion, the whole class is collectively engaged in a mathematical discourse, usually launched by the teacher, explicitly formulating the theme of the discussion. For instance, after problem solving sessions, the various solutions are discussed collectively, but also, it may happen that students' written texts or other texts are collectively analysed, commented, elaborated. Very often, and sometimes explicitly, they are real mathematical discussions, in the sense that their main characteristic is the cognitive dialectics, promoted by the teacher, between different personal meanings and the mathematical meaning related to specific signs (most of the times belonging to mathematics practice) (Bartolini Bussi 1998). The role of the teacher is crucial, in fact the evolution of signs, principally related to the activity with artifacts, towards mathematical signs, is not expected to be neither spontaneous nor simple, and for this reason seems to require the guidance of the teacher. It is quite difficult to fully explain the nature of this "guidance." It cannot be completely assimilated to what is called the *institutionalization* process (Brousseau, 1997), although it is compatible with it. Accordingly the main objective of a teacher's action in a mathematical discussion is that of fostering the move towards mathematical signs, taking into account individual contributions and exploiting the semiotic potentialities coming from the use of the particular artifact.

The semiotic analysis carried out above highlighted a system of relations between the artifact and different signs, and leads one to assume the presence of a particular categories of signs. The criteria of this classification refer to the status of these signs in the evolution process, as it is foreseen in the didactical cycle, from signs related to the activity with the artifact to mathematical signs that are to be related to their standard mathematical meanings as it is shared by the mathematicians community, and may be expressed by a mathematical definition.

Different signs can be identified in the evolution process assumed above generating what can be called a semiotic chain, similar to a chain of signification as Walkerdine (1990) described it: "Producing a particular chain of relations of signification, in which the external reference is suppressed and yet held there by its place in a gradually shifting signifying chain" (p. 121). Further elaborated by Hall (2000) and Presmeg (2006), the notion of *semiotic chain* (or chaining) is also crucial in our theoretical framework, but with a particular emphasis on the role of an artifact in triggering the semiotic process. In fact, according to our approach, such a semiotic chain moves from highly contextualized signs, strictly related to the use of the artifact, to the mathematical signs that are objects of the teaching–learning activity. Besides the category of mathematical signs, we have identified two more categories characterized by their function in the process of semiotic mediation.[13]

Categories of signs

As said, previous semiotic analysis leads one to assume the presence of particular categories of signs, according to an increasing degree distance from the reference to the artifact, i.e., moving from an explicit reference to the use of the artifact towards the mathematics context. As the presence and the status of the signs belonging to the different categories vary in the evolution process, they can be used as an index of the move from personal sense to mathematical meaning. There are basically three main categories: artifact signs, pivot signs, and mathematical signs.

Artifact signs refer to the context of the use of the artifact, very often referring to one of its parts and/or to the action accomplished with it. These signs sprout from the activity with the artifact, their meanings are personal and commonly implicit, strictly related to the experience of the subject, they may be related to what Radford (2003) calls *contextual generalization*, namely a generalization which still refers heavily to the subject's actions in the precise context of the task. Stretching the terminology of Noss and Hoyles (1996), one could speaks of *situated signs,* referring to the term they used "to describe how learners construct mathematical ideas by drawing on the webbing of a particular setting, which, in turn, shapes the way the ideas are expressed" (p. 122).

Contrary to what could be expected, there might happen that no shared meanings for artifact signs emerge, but the direct reference to a common experience may assure the possibility of negotiating a shared meaning within the class community. Although it may occur that signs spontaneously emerge, certainly signs appear and meanings come to be expressed according to specific needs of the context, in particular under the stimulus of specific tasks:

- When the task is asked to be carried out in pairs; in fact, working in pairs may generate the need of communicating and consequently the production of signs referring to the use of the artifact.
- When a written report is asked to be elaborated on during the solution of a problem; for instance this occurs when pupils are asked to fill work sheets.
- When pupils are asked to prepare a delayed written report on what has been done: summarizing the content of a discussion, make explicit their doubts, and so on.

The category of artifact signs includes many different kinds of signs, and of course, non verbal signs such as gestures or drawings, or combination of them (Arzarello 2006). Although our examples will not consider the case of gesture, we want to stress that what we say is consistent and for some aspects complementary to the analysis of Arzarello. In fact, gestures are often a prelude to verbal expressions, mainly in the case of absence of pre-existing verbal elements (Goldin-Meadow, 2000). The artifact signs because of their direct reference to the artifact and its use are mainly used for identifying or focussing on a particular aspect of the (use of the) artifact to be related to the mathematical meanings that are the object of the intervention. They are the basic elements of the development of semiotic process centred on the use of the artifact and finalized to the construction of mathematical knowledge.

Mathematics signs refer to the mathematics context, they are related to the mathematical meanings as shared in the institution where the classroom is (e.g., primary school; secondary school) and may be expressed by a proposition (e.g., a definition, a statement to be proved, a mathematical proof) according to the standards shared by the mathematicians community. These signs are part of the cultural heritage and constitute the goal of the semiotic mediation process orchestrated by the teacher.

Through a complex process of texture the teacher construct a semiotic chain relating artifact signs to mathematics signs, expressed in a form that is within the reach of students. In this long and complex process, a crucial role is played by other types of signs, which have been named *pivot signs*. As the following examples will illustrate, the characteristic of these signs is their shared polysemy, meaning that, in a classroom community, they may refer both to the activity with the artifact; in particular they may refer to specific instrumented actions, but also to natural language, and to the mathematical domain. Their polysemy makes them usable as a pivot/hinge fostering the passage from the context of the artifact to the mathematics context. Very often they mark a process of generalization, this is the case of generic expression like <object/s> or <thing/s>, as well terms of the natural language that have a correspondence in the mathematical terminology. Their meaning is related to the context of the artifact but assumes a generality through its use in the natural language. Sometimes they are hybrid terms, produced and used within the class community,[14] they are meant to express a first detachment from the artifact, but still maintaining the link to it, in order not to loose the meaning.

The diagram of Figure 28.1 can be re-elaborated according to this classification as shown in Figure 28.2.

It is worthwhile to note that the same word, as soon as it is considered a sign, may correspond to different categories, according to its use in the semiotic activity; for instance, the word "function," may be used according the vocabulary of natural language and refer to the context of the artifact, in this case it will be considered an artifact sign; it may achieve a certain autonomy from the artifact and become a pivot sign; finally it may be used explicitly referring to a mathematical definition and in this case it is classified as a mathematics sign.

Semiotic mediation in the classroom

The theoretical framework introduced above can be used to analyse different teaching experiments carried out in the past; all of them were differently inspired by a Vygotskian approach,

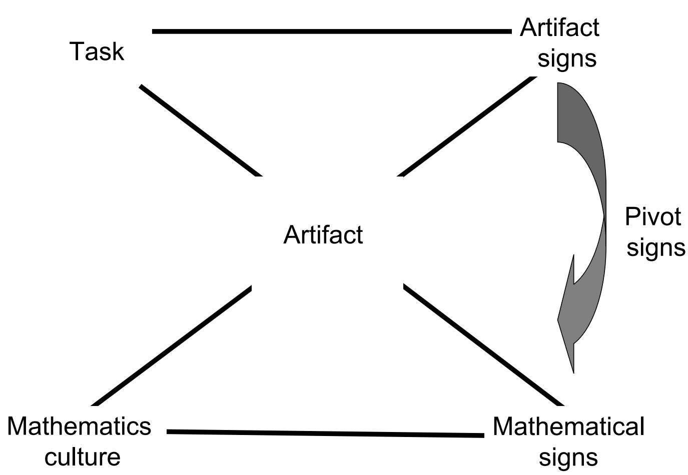

Figure 28.2 Artifacts and signs.

but involved students of different ages and mathematical ideas from different mathematical domains. We present some examples drawn from two of them, these examples are to be considered paradigmatic and aim at illustrating and clarifying the theoretical notion of *tool of semiotic mediation*. More examples can be found in previous articles (Bartolini Bussi & Mariottti, 1999; Mariotti, 2000, Mariotti, 2001; Cerulli & Mariotti, 2002, 2003; Cerulli, 2004; Bartolini Bussi, Boni, Ferri, & Garuti, 1999; Bartolini Bussi, Mariotti, & Ferri, 2005).[15]

The first example concerns a well know cultural artifact: the abacus. Its origin is closely related to the history of counting, it had a primary part in the history of mathematics, never the less, in some part of the world the *abacus* in its various forms of counting boards, is still in use in everyday life (e.g., China, Japan, Russia). The abacus is also familiar in school practice and in some respect is to be considered a traditional didactical aid. The second example concerns a modern artifact belonging to the set of products generated by new technologies in the case of Dynamic Geometry Environment: *Cabri*[16] (Mariotti, Laborde, & Falcade, 2003; Falcade Laborde, & Mariotti, 2007). The two artifacts present interesting differences. In the case of the abacus, the analysis providing its semiotic potential is strongly dependent on the historic reconstruction of the evolution of its use into the decimal positional notation of numbers. This makes the example of abacus paradigmatic: it may be considered a model for the analysis of other artifacts that share its historic relevance: compass, perspectographs, linkages for tracing curves, and so on. In the case of Cabri, like in the cases of other new technology's products, the analysis providing its semiotic potential depends on what have been called the "computational transposition" (Balacheff & Kaput, 1996; Balacheff & Sutherland, 1999), meaning that the relationship between a certain mathematical domain and a computer-based artifact may depend in a complex way on choices in design and implementation. The actual implementation of a computer-based artifact requires decisions at the programming level taking into account the constraints of the operating system of the machine, the specificities of the programming language and of the related representations.

The exposition of the two examples will follow the same structure. As a consequence of the previous discussion, we will start with presenting the semiotic potential offered by the artifact, in the case of the microworld of Cabri, this analysis will be limited to specific functionalities related to the use of specific commands. Subsequently, we will analyse protocols coming from the data collected during the teaching experiments, highlighting the emergence and the evolution of signs. Some remarks on the action of the teacher will be added, but not fully discussed. A fine grain analysis of teachers' actions is still in progress, and until now only few examples have been described (see, for instance, Bartolini Bussi, 1998; Mariotti & Bartolini Bussi, 1998; Mariotti, 2001, Falcade, 2006).

THE CASE OF THE ABACUS

Analysis of the semiotic potential: The abacus and the positional system of representation of numbers

The abacus, in its various forms of counting board (Menninger, 1958, p. 297), is a well-known cultural artifact. Its origin is closely related to the history of counting and of record-keeping. The tokens, either clay tokens or collections of pebbles or tally sticks or the like (p. 223), had two main functions: they served as counters to calculate quantities of goods; they were mnemonic devices used to store data, the aim being always to have as many items (tokens, pebbles, notches) as the "objects" (e.g., animals, days for lunation, pieces of food). Counting boards with pebbles or counters were used later to make computations (especially additions and subtractions). The basic idea was to break the collection of unity-pebbles down into groups and to handle groups instead of individuals. Each group was *represented* by a pebble (sign), that in order to be distinguished from a unity-pebble, was put in a different position on a board or a tablet divided into strips or columns.

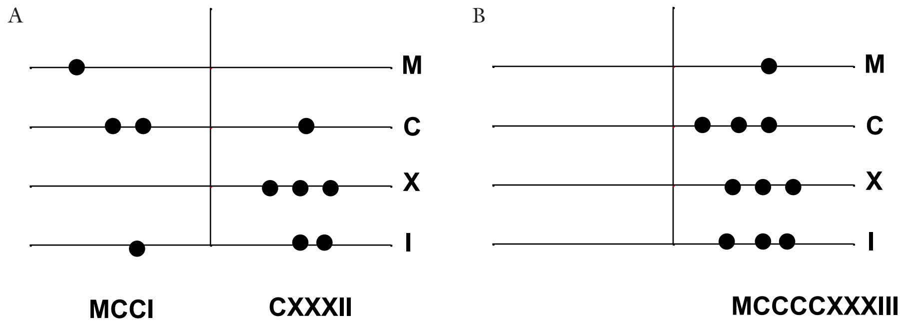

Figure 28.3 A. Roman numerals are represented on each side of the counting board. B. The pebbles are pushed from the left side over to the right side to represent the answer to the addition problem.

The number of pebbles in each group was defined by the base, usually 5 or 10, as a reminder of the ancient numbering by fingers of one or both hands. For centuries, this strategy was used to make computation. For instance, to make addition, the two numbers were represented together in the same abacus, before grouping the counters (if necessary) in each strips column. For instance, to add the numbers 132 and 1,201 (that in Roman numerals are written as CXXXII and MCCI), the first step is to represent them on each side of the counting board (see Figure 28.3a and Figure 28.3b).

The next step was to push the pebbles from the left side over to the right side so that they are all together to represent the answer to the addition problem. Then the result was written again by means of numerals (Menninger, 1958, 319ff). So, in computation, people used a mature place value system of representation for numbers (where also empty strips or columns were allowed) whilst in writing different kind of signs were used (p. 223).

The above figures refer to computations similar to the ones of the famous plate taken from the Margarita Philosophica of Greigor Reisch (1503; see Figure 28.4), where calculators using the counting board and written numerals are represented (Menninger, 1958, p. 350).

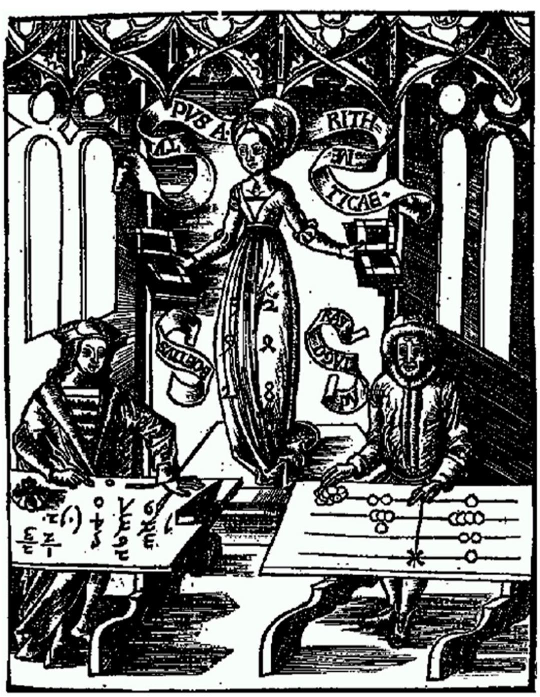

Figure 28.4 Plate taken from the Margarita Philosophicaof Greigor Reisch (1503), representing calculators using the counting board (right) and written numerals (left).

It may be surprising that the use of the abacus, where beads have values according to their position, did not produce everywhere at the same time a positional system of representation of numbers. In Europe, for centuries, the ways of writing numbers and of representing numbers on abaci (or counting boards) developed independently from each other. Only in the Middle Ages, thanks to the mediation of Arab culture, was the way of representing numbers by means of only 10 ciphers and of positional conventions introduced to Europe. Together with written algorithms, this allows computational operations to be made on a sheet of paper without use of an abacus.

Today, one can say that the abacus has the potential to refer to arithmetic meanings like the general polynomial representation of numbers:

$$a_n x^n + a_{n-1} x^{n-1} + \ldots\ldots + a_2 x^2 + a_1 x + a_0.$$

The historical process of construction of arithmetic meanings left traces in the artifact-signs that originated from the ancient practices and, in some cases, became true mathematical signs. For example:

- the very name of *abacus*, coming from the Greek word *abax* (from Hebrew *abaq,* i.e. “dust” or “sand used as a writing surface”);
- the very name of the activity (*calculus*) to hint at the pebbles (*calculi*, in Latin);
- the very name of *zero* to hint at the empty state in some column (Menninger 1958, p. 401).

The loss of materiality allows one distance from the empirical facts. This deserves attention, as it might be applied to other cases.[17] The transition from abacus to paper (with the need to introduce the special sign “0” for the empty groove) is

> to shift from a gestural medium (in which physical movements are given ostensively and transiently in relation to an external apparatus) to a graphic medium (in which permanent signs, having their origin in these movements are subject to a syntax given independently of any physical interpretation). (Rotman, 1987, p. 13)

A concrete abacus is a *primary artifact* in Wartofsky’s sense (1979): the need of counting and of record-keeping is historically documented from very ancient ages in the production of the means of existence. The way of using abaci for counting, record-keeping and computation, the iconography, the written representations (see Menninger, 1958, p. 297 ff.) are *secondary artifacts.* A secondary artifact is also the *graphic abacus* that is used to illustrate numerals

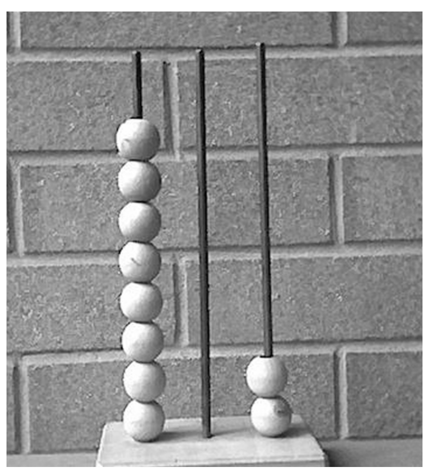

Figure 28.5 A spike abacus similar to the one used by children.

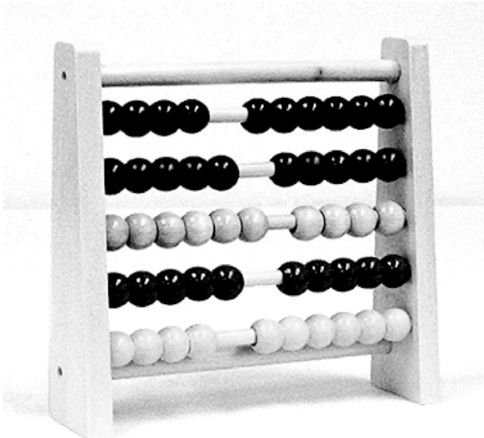

Figure 28.6 A Slavonic abacus similar to the one used by children.

and operations, with reference to the *spike abacus*, made by vertical wires and beads to be strung. The polynomial representation of integers in base 10 (or, later, in a base) is a *tertiary artifact*. From an historical perspective, the positional system is not "embedded" but rather an important yet unexpected by-product (and even a late one) of the century use of abaci in computation.

The abacus considered in Figure 28.5 has explicit reference to the position of beads. There is another abacus, the so-called *Slavonic abacus* (Figure 28.6) derived from the Russian *ščët*, where 10 horizontal wires contain 10 beads each. Grouping beads by 10s speeds up and controls counting and fosters the investigation of patterns.

Using any abacus in counting or reckoning tasks makes utilization schemes emerge. Each type of abacus produces special gestures, sometimes explicitly taught, as illustrated in Figure 28.7, taken from a Chinese textbook for first graders, where the action of the various fingers is shown.

In the following, we shall describe separately the utilization schemes for the Slavonic abacus and for the spike abacus, for the sake of clarity.

The Slavonic abacus

When tasks concerning *counting* and *storing data* are given, it may be used as a scoreboard, where each bead represents a unit. The earliest utilization scheme is:

Scheme 1*: counting*, including two sub-schemes,
Scheme 1a: one-to-one correspondence: to move a bead for each item to be counted,
Scheme 1b: counting all,
Scheme 2: (10s) count each full wire as 10 and the remaining beads as units.

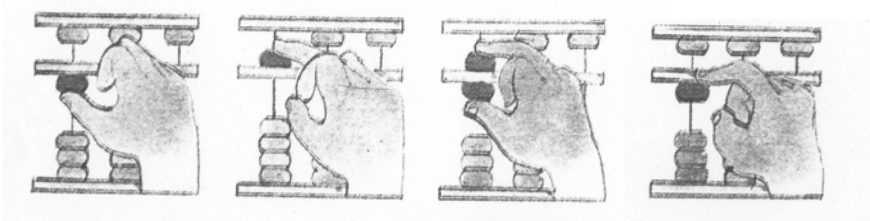

Figure 28.7 Instructions from a Chinese textbook for first graders.

This use of abacus is consistent with the additive number system (one mark for each item). This use is consistent also with the grouping strategy (grouping together 10 items to handle them as a whole), that is one of the elementary steps of the decimal positional system. However, this grouping strategy was used also before the decimal positional system in number representation was adopted: consider for instance the Roman system, where a special sign (X) was used to denote a group of 10.

When the items to be counted are given by two different collections, the number of beads given by the first addend is moved first, before acting on the number of beads given by the second addend.

In this use, typical artifact signs are <beads> or <wires> and <moving>, such words are used by the pupils and the teacher directly referring to the objects and the actions involved in the use of the abacus. Other examples in the classroom setting are discussed below.

The spike abacus

Scheme 1 works also with the spike abacus, when a very small number is involved and no convention about position is known. In this use, typical artifact signs are <beads> or <wires> and <stringing>. The comparison (or the simultaneous use) of the Slavonic and of the spike abacus makes a potential conflict emerge: whilst the Slavonic abacus works drawing on one-to-one correspondence only, whichever is the size of the number, the spike abacus works drawing on a mixed convention: one-to-one correspondence for "small" collections and positional value of beads for larger collections of objects to be counted.

There is no natural path towards the use of the abacus according to the positional system; the only exposure to written and oral numerals in everyday practice (e.g., the numerals of the calendar pad or price labels) is not enough to suggest the position convention, unless the subject takes part in suitable practices. If only nine beads (one less than the base of the positional system) fit in a wire, the mismatch between the grouping suggested by the abacus and the one suggested by the oral number litany paves the way towards a different use of the beads, which considers also the position. However, the hints must come from an expert (as it happens in the social practices of school instruction). The conventional use of abacus determines another utilization scheme:

Scheme 3 (exchange): 10 beads on the first wire from the right are exchanged with a bead on the second wire; 10 beads on the second wire are exchanged with a bead on the third, and so on.

This scheme comes from the social use of abacus. It inhibits the possibility of having additive system, because the value of a bead depends on the wire (i.e., the position) where it is strung. The exchange scheme is two-way: it is possible to exchange 10 beads of the first wire with one of the second (as it happens in additions with carrying) or vice versa (as it happens in subtraction).

The individual process of constructing utilization schemes which are consistent with the positional system is interlaced with the semiotic process of appropriating the oral number sequence and the written numeral sequence and, above all, with the appropriation of the social scheme "exchange." Other utilization schemes emerging in the classroom setting will be discussed in the following.

The previous analyses describe the semiotic potential of two different kinds of abaci and may constitute the base for the design of a teaching sequence according the general structure of the didactical cycle. The early tasks include counting, reckoning and representing numbers. Representation tasks include: *representing* by means of the abacus (either a Slavonic or a spike one) a number given in oral or written form, *reading* the number and *writing* the number represented by the abacus, and so on. In this way, children are introduced and may appropriate the meaning of the polynomial representation of numbers, in particular they are induced to distinguish between a number and its representation. The use of the abacus to solve counting, reckoning and representation tasks creates a setting where semiotic activity is encouraged with an intense production of signs, where both artifact signs and mathematical signs sprout, live, and develop.

TEACHING EXPERIMENTS

Several teaching experiments have been carried out in Italy by the first author (and collaborators) in all primary school grades concerning the function of abacus as a tool of semiotic mediation. The main activities structuring the didactical cycle were the following:

Activities with abacus: pupils were faced with tasks (e.g., counting, reckoning and, with the teacher's help, also representing numbers) to be carried out with the artifact. These tasks were initially solved in social situation (small group or whole class discussion) with the production of gestures and verbal signs.

Individual production of signs (e.g., drawing, writing). Pupils were asked to write individual reports (in writing and with drawing, if needed) on their own experience and reflections, including doubts and questions arisen. This activity is centered on intentional semiotic processes, i.e. the production and elaboration of signs, related to the previous activity with abacus. It allows the teacher to have a collection of signs, produced by the students. Most of them showed a direct reference to the manipulation of the artifacts.

Mathematical discussion about the signs that have been produced individually: in this step the teacher had the responsibility of orchestrating the polyphony, where the voices (represented by the signs produced by the pupils) were coordinated with the voice of the mathematical culture (witnessed by the teacher).

Written dialogue. Also individual tasks requiring the use of abacus were given, with the request of producing a written solution (with drawings, if needed), as a base for further mathematical discussions. During the individual tasks, the teacher sometimes interacted with the pupil, by means of a written dialogue, i.e., writing a short comment on the worksheet to ask for explanation, to re-launch the process and the like.

This cycle was not rigidly fixed and was open to changes, according to the particular conditions of activity. Some examples will be discussed in the following. They come from different teaching experiments carried out in different grades, although based a didactical cycle with the same structure. We obtained similar results in the different classrooms, so that the selected protocols can be considered good representatives of pupils' performances at different stages of the process.

A chain of artifacts in the first grade

The sequence was structured into two main phases, represented by two different abaci: the Slavonic abacus and the spike abacus. The teacher decided to postpone the introduction of the spike abacus to avoid a purely conventional agreement on the different meanings (units, 10s, 100s, and so forth) of the wires, according to position, until the pupils showed to master the one-to-one correspondence between the items to be counted and the beads of the Slavonic abacus. She was afraid (on the base of previous experience with first graders) that a mechanical introduction of the exchange rule might have hidden other important features of numbers (e.g., patterns) not related to their representation.

The Slavonic abacus

In this classroom (23 pupils, 13 girls and 10 boys) in the first months of school, the pupils had been exposed to oral and written numbers taken from everyday contexts (e.g., calendar, courtyard games). Counting had been practiced in many situations. After 3 months, a Slavonic abacus with three wires had been introduced for taking note of the result of counting and for practicing with numbers. The practice with this abacus was aimed at mastering the counting process (that showed still uncertain), at comparing numbers and investigating some patterns (e.g., the complement to 10). The counting process was speeded with the line grouping by 10. As expected, intense production of signs was observed. Expressions like <a 10 beads line> or <a 10 line> or <a full line>, that can be classified as artifact signs, were produced, shared, and used everyday; the abacus was drawn very often (also in tasks of real-life drawing). This detailed observation of the artifact aimed to foster the process of instrumentalisation, when the user has to become familiar with the components of the artifact. Other activities concerned reading, writing, and reciting numerals coming from everyday practice without calling attention on the positional convention. Two-digit numbers (e.g., calendar, present pupils) were simply registered on the Slavonic abacus, filling one or more wires or, as the pupils said <10 lines> (in Italian <*linee da dieci*>). After some practice, the shared knowledge was summarized by the following text, written as a result of a collective discussion: *In the numbers with the 10-line, there is the word "teen." When there are10-lines, the number has two digits. The 10-lines are written on the left*. This text is hybrid and clearly shows the product of the teacher's action of using the signs produced by the pupils (e.g., 10 line) to refer to either the mathematical name (the suffix "teen") or the early idea of positional code ("on the left"). In other words, the sign "10 line" has been used as a pivot sign, whose polysemy refers to both the artifacts and the mathematical domain.

The spike abacus

On March 13, the teacher attacked the meaning of a positional system in a direct way, and tasked the students with interpretation: a spike abacus with the number 13 was shown (for the first time) on the teacher's desk, being the choice of the number (13) the same of the day (see Figure 28.8). The written task was the following:

The first task: Which number is it? Copy the object, answer in writing and explain why.

Nearly all the pupils, at the beginning, after having sketched correctly the abacus, answered "4," thinking of it as a scoreboard, according to the one-to-one correspondence scheme, practiced with the Slavonic abacus. Only one girl recognized (with hesitation) the right number. The written interaction between the girl (G) and the teacher (T) is literally translated below:

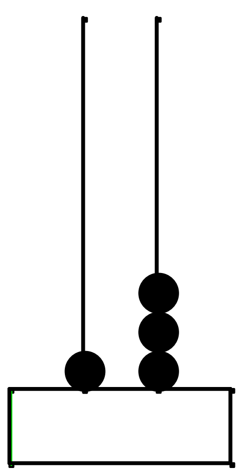

Figure 28.8 The first task on the spike abacus.

Excerpt 1

G: Like for the days I have understood that you every morning do with the calendar. It is 31.
T: Read better.
G: It is 13.
T: Why?
G: I had thought that 1 seems 10 and 3 seemed the 13 that number has told me 13.

Five other pupils, after an individual written interaction with the teacher, succeeded in writing the right number. For instance, a girl (S.) wrote:

Excerpt 2

S: 4
T: No
S: At left there is the number 1 and at right the number 3. this number is called 13.
T: If it is so, explain well why.
S: Because first there is the left wire and there is the number 1 and in the other wire there is the number 3 and together the number 4 forms but if you add a 10 line, it becomes 13.

This protocol, similar to many others, shows a snapshot of a developmental process. At the beginning, S interpreted the spike abacus like a Slavonic abacus, where the one-to-one correspondence and the counting all schemes were used. When the teacher evaluated the answer negatively, she changed her mind and produced a different (correct) answer. However, the interpretation linked the two kinds of abaci: the 1 bead was not to be exchanged (conventionally) with 10 beads, but was rather a reminder or a representative of a 10-line (the 10 scheme). In her mind, S. was linking the spike abacus, the new artifact, with the more familiar Slavonic abacus. This interpretation was shared by most pupils as shown in the solution of the following task. The second task was given immediately after the first one. The teacher read aloud some different solutions of the first task without commenting them and, pointing to the abacus and pretending for a while that she did not know the solution, asked: *What ever number is it?* After a short pause, she continued: *I shall tell you: it is thir-teen,* slowing down the pronunciation to create a hiatus to emphasize the structure of the word. Then she wrote on the blackboard: *It is the number 13* and asked again: *Explain why it is so.* Giving the solution, but asking for an explanation, the teacher engaged the pupils in a semiotic activity: she invited the pupils to interpret mathematical signs (the oral and the written numerals) referring to the use of the artifact (the abacus with the given beads).

The pupils copied on their worksheet the writing on the blackboard, drew again the abacus and wrote their own explanations. The pupil's performances changed dramatically: all the pupils, but five, succeeded in giving appropriate justifications. Most pupils explicitly referred to the utilization scheme of the Slavonic abacus (the 10 scheme), where the reference to the <10 line> was explicit. For instance, *it is thirteen because there is a 10 line and a 3 and if the 10 line join 3 becomes 13.* Some pupils showed a transition to a conventional utilization scheme (the exchange scheme). For instance, *because it has a bead on the left and three beads on the right and the one of the left is worth 10 and the one on the right is worth 3.*

A more articulated example is the following:

Excerpt 3

A. wrote: It is thirteen because in a wire there is 1 that is worth 10 and in the other there is the number 3 and if you put them together they form the number 13.

In the following discussion, she added: *I have understood. The Slavonic abacus becomes a spike abacus,* and, while speaking, she rotated the Slavonic abacus in order to transform the upper row into the first column on the right, i.e., the unit wire.

The influence of the Slavonic abacus appeared to be very strong. It is interesting to observe that in the following lessons, a new shared sign emerged in the classroom: the pupils, instead of saying, "to exchange 10 beads with a bead on the other wire" (that is the traditional way of presenting the convention), preferred to say. "to compress (Italian stringere) 10 beads in one bead of the other wire."

Artifact signs like <wire>, <bead on the left>, and <bead on the right> refer to the positional distinction of values and for this very reason they are candidates to evolve towards mathematical signs.

In the following discussion, the teacher reconstructed the link between the Slavonic abacus (and its 10 beads wires) and the spike abacus. She introduced the practice of stringing beads one at a time in the right wire of the spike abacus, up to nine; for the tenth, the right wire was "emptied" and only one bead was strung on the left wire.

A few days later, a third task was given; the number 10 was represented and the same question of the first task was repeated: *Which number is it? Copy the abacus, answer in writing and explain why.* The task was more difficult because of the interpretation of the empty wire. A pupil wrote:

Excerpt 4

> The abacus is used for making numbers. On the left there are tens and on the right there are units. In the abacus there is the number 10. When the beads are 9 we empty [the wire] to put the tens. They are on the left.

In his drawing (Figure 28.9), a collection of nine beads appears and is placed horizontally, with an evident reference to the original orientation of the first wire in a Slavonic abacus. It seems that the pupil in at an intermediate stage, still needing to maintain the link between the old and the new artifact.

From then on, the semiotic activity continued systematically: every day, the calendar date of the day and of other meaningful numbers (e.g., the temperature) was represented on the spike abacus and drawn and written in words and in 2-digit numbers on the individual notebook.

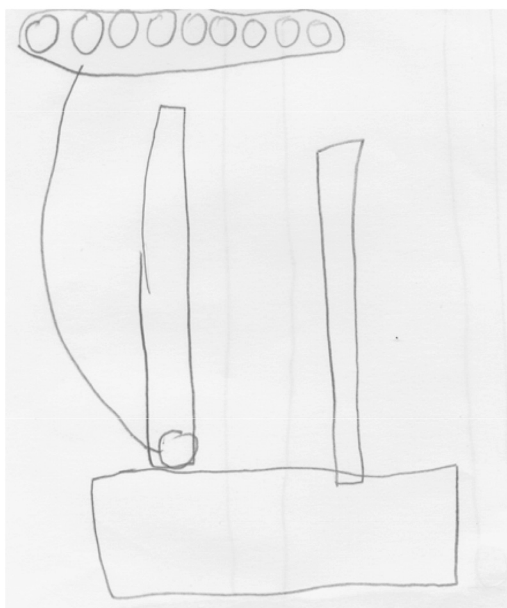

Figure 28.9 The reference to the Slavonic abacus.

Pivot signs in the second grade

The analysis of the previous example showed the emergence and the evolution of artifact signs as well as their close relationships with specific features of each artifact they derive from. Moreover, in the same sentence, both artifact and mathematical signs appear; such a hybrid character of the sentence witnesses that the expected evolution is in progress. In the following example, the hybrid character is condensed into a single word, that combines elements coming from the artifact and from the mathematics. It was reproduced by a girl to solve an addition problem, drawing on the signs shared in mathematical discussions. The protocol refers to another classroom, where only the spike abacus had been used, within a similar didactical cycle. The pupils had produced and shared artifact signs similar to those emerged in the other classroom. At a certain point (second grade), the following task was given:

> A 7-year-old child has made the following calculation:
>
> $$37 + 15 = 52.$$
>
> He tells that he has used an abacus. According to you, how did he make? Write your explanation and if you wish illustrate it with drawings.

The task, given by means of mathematical signs, explicitly called into play the abacus that had been used from the first grade on. Student B. proposed the following solution to the problems. In the transcript, we differently underlined some examples of artifact signs and mathematical signs.

Excerpt 5

B.: I take an abacus in which to make the number 37, I string 7 units in the units column and 3 tens into the tens column. Then in the same abacus I arrange the second number (the 15), but while I put the 5 units I realize that there is not enough room for them then I move to the tens to see whether I can solve the case, I string 1 10 of the 15. Then I come back to the problem of the 5 units, I use a bit of my brains, I string 2 of them to free my hands a bit, but still I have 3 of them left in my hands. I make an exchange: I exchange 10 units with one bead-10 (<pallina-decina>) that I take from the basket and I string it in the wire of tens and empty the 10 units in the basket. Surprise now there is room on the wire for the 2 beads left (that I have in my hands), string them in and the result is 52.

The exchange scheme is explicitly recalled with the sign <exchange> together with the sign <bead-10> (in Italian, <*pallina-decina*>). The drawing accompanying the text is not a realistic drawing of the abacus, but a sketch of the graphic abacus where the shift-exchange of 10 beads from the unit wire to the 10 wire is dynamically represented with arrows and where the position value is represented with labels (U means units, DA means tens; see Figure 28.10).

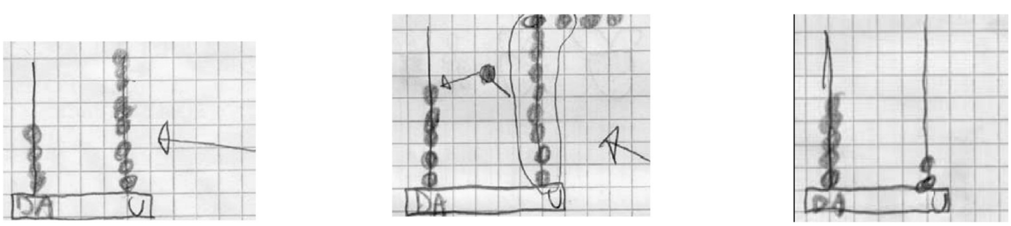

Figure 28.10 B's drawings.

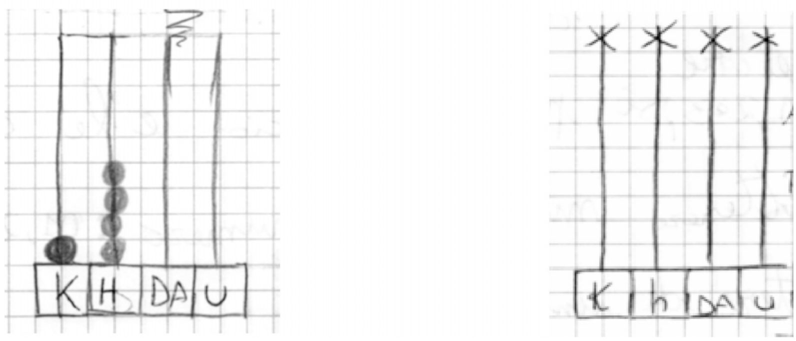

Figure 28.11 M's drawings.

The generation of a new mathematical meaning in the third grade

In a third-grade classroom, pupils were asked to explain to an imaginary middle age interlocutor the positional system of representation of numbers, using words, drawing and even artifacts if needed. Most of the pupils spontaneously decided to use the abacus that had been used in the classroom in the first and second grades. In the following short excerpt, a pupil (M.) explains how to represent 1,400 to show the delicate role played by zero. The excerpt contains also two drawings.

Excerpt 6

M.: I try to show you the number 1400. There can be only 9 beads, do you know why? Because if you put 10, you have to carry a number and go to the next column. No there is no abacus where to put 10 beads. The empty position means the digit 0. (It might exist, but you should count in base 11.)

The first drawing (see Figure 28.11 on the left) represents correctly the number, but M. wished to give more details to the imaginary interlocutor and to explain the meaning of base. He then prepared a new sketch (Figure 28.11 on the right). He probably drew too fast and the lines representing the wires were too long. After a check, he recognized his mistake and crossed the wrong extra pieces of the wires. Yet, after being drawn, the graphic abacus with the too long wires acquired a new status and generality and suggested to M. a completely new possibility, that of using a different base where the exchange scheme could be used for 11 beads. This insight was sudden and, after the conclusion of his reasoning. M. felt compelled to add this comment in brackets at the end. The semiotic activity intentionally induced by the teacher put M. in a very favorable situation to conceptualize the idea of base of the positional system in a much more general way, taking the distance from the only known case that was the base 10. The abacus, hence, proved to have become for M. a tool of semiotic mediation for the idea of a whichever base was used for the positional system of representation of numbers.

THE CASE OF CABRI

The multiform reality of variation has become in mathematics an object named *variable*, while particular relationships of dependency between variables has become a mathematical object named *function*. In the case of functions, the history of co-variation may be traced back at least at Descartes' geometry (1636) and at the later development of the so-called organic geometry, related to the use of drawing instruments (Bartolini Bussi, 2001). Nevertheless, the following example will not refer to historic sources, but rather illustrates what can be done with an artifact belonging to the new computer-based craft, that is the DGE Cabri.

According to the previous discussion, we first elaborate on the semiotic potentialities of the chosen artifact; in particular some of its commands (named *tools* in the following) will be analyzed in order to make explicit the epistemological relation between their functioning and their use and the mathematical notion of function, in particular in relation to its crucial meaning of co-variation, i.e., a relationship of dependency between variations.

Analysis of semiotic potential: Cabri and the notion of function

Because a full analysis of the semiotic potential of the artifact Cabri would be too complex and certainly beyond the scope of this section, we limit it to explain some key elements that will be related to the mathematical content in focus, i.e., the mathematical notion of function. Elsewhere other partial analyses were carried out in relation to this notion (Falcade et al., 2007) and other mathematical notions, see for instance (Mariotti 2000, 2001) in the case of the notion of theorem.

The basic functionalities of a DGE such as Cabri concern the domain of graphics and are concerned with *construction* and *motion* (usually referred to as *dragging*): the former allows one to generate graphic traces on the screen by the use of a number of construction tools (Point, Line, Perpendicular line),[18] the latter allows one to make these traces move. The way constructions and motion are accomplished is determined by the functioning of the tools of the microworld, that in our terminology is an artifact, and by the use of its different tools that the user do, in our terminology its utilization schemes. The fundamental characteristic of a DGE, relating the construction and the motion functionalities, concerns the stability of the graphic product of a construction under the effect of the Dragging tool. That means that whatever property is defined by a construction tool it will be maintained in the motion by dragging. A Cabri object (i.e., a drawing) on the screen can be moved using the Dragging tool, activated through the mouse, preserving its defining properties.

There are two main kinds of motions, direct and indirect, corresponding to two different utilization schemes of the Dragging tool.

The direct motion scheme occurs when a basic element (e.g., a point generated by the point tool) can be dragged on the screen by acting directly on it: the motion of a basic element represents the variation of this element in the plane; the correspondence between *motion* and *variation* constitutes the main characteristic of any DGE and of Cabri in particular, so that the geometric notion of *generic point* of the plane is nicely represented by a *basic point* (the combination of a point on the screen and its possibility to be dragged, i.e., its possibility of being freely moved on the screen). Consistently, a *point on an object* represents the variation of a point within a specific geometrical domain, a line, a segment, a circle, and the like; in other words it represents the geometric notion of *point belonging to figure*, i.e., a sub-set of the plane.

The indirect motion scheme occurs when a *construction procedure* is accomplished, i.e.. when a drawing is obtained through the sequential activation of construction tools; in this case, dragging the basic points from which the construction procedure originates, will determine the motion of the new elements obtained through it; this motion preserves the geometrical properties defined by the construction and consequently the geometric properties of the image.

As a consequence, using the Dragging tool according to the combination of the two schemes, the user may experience the combination of two interrelated motions, the free motion of basic points and the dependent motion of the constructed points; in other words, the use of dragging allows one to feel *functional dependency* as the *dependence relationship between direct and indirect motion*. Hence, the *Dragging tool* (artifact) may be considered a *sign* referring to the mathematical notion of *function as co-variation* between dependent and

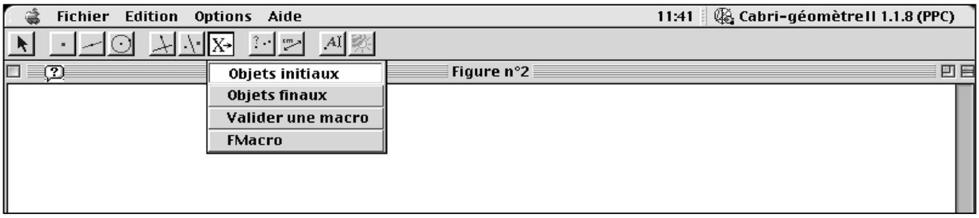

Figure 28.12 The macro tool as it appears on the screen.

independent variables; similarly a *construction* procedure may be considered a *sign* referring to the relation of functional dependency between variables.

The artifact sign "point (freely) moving on screen" refers to the notion of a (independent) variable; the artifact sign refers to its movement, that is to its space/time variability. The didactic aim, consistent with the previous semiotic analysis of the artifact, is that of preserving the content of this sign in the subsequent evolution: from the artifact sign (moving point) to the mathematical sing (variable).[19]

The sequence of tools employed in a construction procedure may be condensed in a single tool through the definition of a *Macro*: the new tool is created following a guided Cabri procedure (see Figure 28.12) that asks for the initial objects, the final objects and finally validates the construction. Each macro can be saved as a new object and given a name in order to be retrieved and used whenever one needs it. A macro is activated providing as input the initial objects and will produce as output the expected final objects. The schemes of utilization of a Macro are related either to the definition of a new Macro or to the execution of an already defined Macro.

When a new Macro is defined, a relationship of dependence is explicitly stated between the initial objects (usually points) and the final objects; similarly the application of a Macro activates a relationship between input and output objects. According to this interpretation the *Macro tool* can be considered a sign referring to the notion of function as an *input/output machine*, where the initial objects and the final objects are respectively the independent and the dependent variables; in particular, the fact that a Macro is expected to be named and saved as a single file evokes the idea of function as a single object, condensing the infinite set of correspondences between the occurrences of the independent variable and the occurrences of the dependent variable.

Movement of points experienced (both visually and kinetically) through the use of the Dragging tool can be materialized through the Trace tool: its use allows display of the track of a moving point, i.e., the *trajectory* of this point. Although the final product of the Trace tool is a static image consisting in a set of points, the use of Trace tool involves time; actually, one can feel time running in the action of dragging, in particular when changing the speed of dragging, but also one can feel time running in the variation of the dependent point. As a consequence, it is possible to grasp simultaneously the global and the pointwise aspect of the product of Trace tool, which can be related to the mathematical notion of *trajectory*: at the same time a sequence of positions of a moving point and the whole object (a curve) consisting in the set of all such positions (see Figure 28.13).

The Trace tool can be activated on any point independently, thus it is possible to generate the trajectory of either the independent or the dependent point or both. Each of the two correlated trajectories appears progressively, meanwhile they are generated point by point, and finally they can be globally perceived as two sets of points: in so doing both the domain and the range of the corresponding function become visible.[20]

The graphic mark appearing on the screen after the combined activation of the Dragging and the Trace tools may act as a catalyst (Falcade, 2006, p. 95) to set off the meanings related to the use of dragging. Using Radford' term, we can say that the image produced by the Trace on the screen objectifies (2003) the meaning related to the act of making the point move, so

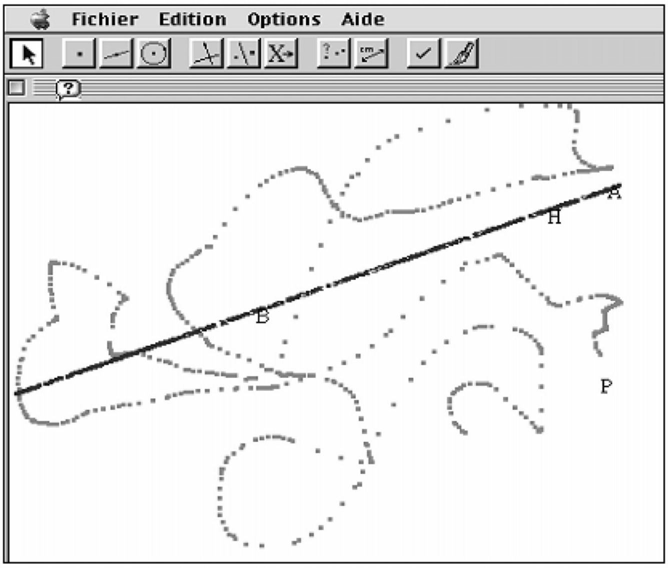

Figure 28.13 What may appear on the screen when using the trace tool.

that both the mark produced through the use of Trace and its name, *trajectory*, may become a sign referring to this act, and in particular to its nature of direct or indirect move. In this sense, the *Trace tool* may be considered as a sign referring to the mathematical idea of *trajectory*, and as such a potential tool of semiotic mediation for the meaning of trajectory.

THE TEACHING EXPERIMENT

Taking into account the main results obtained from the analysis on the correspondence between some Cabri tools and meanings related to the idea of function, a sequence of activities was designed and implemented in class, centered on the use of certain Cabri tools as tools of semiotic mediation. The research project involved four 10th-grade classes (in France and in Italy); the students were 15-16 years of age.

As far as the mathematical content is concerned, the teaching sequence was structured into three main phases. Our examples will be limited to the first activity that was expected to play a key role and become a reference situation; however to have a global idea of the rationale of the teaching experiment we give a brief account of whole sequence:

At the beginning, students were asked to explore the effect of an unknown macro, labeled "Effetto 1"; variation and co-variation, as well dependency, emerged in relation to the motion of points dragged on the screen. Collective discussions referring to the experience with Cabri were carried out with the aim of formulating a tentative definition of function that at the beginning was limited to what were called geometric functions. Such a definition (as well as that of image, domain, range, co-domain) was collectively constructed in the classroom and individually re-elaborated in written texts to be produced as homework. In this phase, the Dragging tool and the Trace tool were the key elements on which the process of semiotic mediation was activated by the teacher.

From geometric functions students moved to numerical functions and approached the problem of representing numerical functions geometrically. Reinvesting the idea of trajectory, as it emerged from the use of the Trace tool and was related to the definition of geometric function, the notion of graph was introduced as a geometrical function associated to a numerical one through a well-defined process. In this part of the sequence, a new tool of semiotic meditation was introduced, that is an excerpt drawn from a text of Euler (1743/1945) (for details see Falcade, 2006).

Finally, the use of the graph of a function was developed as a means to solve problems. The whole sequence was carried out in approximately 2 months.

First teaching session

Let us describe the scenario of the first teaching session that was carried out in the computer laboratory. Students were grouped in pairs and had to produce a common written answer on a worksheet. In the first task, the students were asked to apply the unknown Macro Effetto 1 to three given points: A, B, and P; they obtained a fourth point H. The first question asked to explore systematically the effect of moving one of these points at the time and to fill in a table explaining what moved and what did not move when dragging each point. The second question suggested to use the Trace tool and asked students to observe and describe the movement of the different points, using the current language of geometry.

In this case, the macro-construction provides point H as the orthogonal projection of point P onto line AB. The choice of this construction was motivated by the need of obtaining a significant visual phenomena on the screen, in particular related to the difference between domain and image (Figure 28.13).

Students easily answered the first question, immediately grasping the difference between direct and indirect motion that remained a reference for the meaning of variable either independent or dependent. According to our assumption, the use of the Trace tool contributed to the emergence of the twofold meaning of trajectory, both as a *globally perceived object* and as an *ordered sequence of points*, as clearly shown in the students' formulations (Mariotti et al., 2003; Falcade, 2006).

The evolution of signs

According to the previous discussion, there are different categories of signs according to their status in the evolution process from personal meanings to mathematical meanings: semiotic relationships are explicitly stated between signs belonging to different categories, for instance, artifact signs and mathematical signs, at the same time a critical role is played by the pivot signs.

The following examples will be limited to the case of verbal signs, both oral and written. Actually, during verbal exchanges, both when students work in pairs and during the collective discussions, gestures are often observable: they prelude to verbal expressions, mainly when verbal elements are not yet available to express what is meant.

According to the main assumption, it was possible to observe the emergence of key artifact signs such as <it moves>, <it does not move>. Moreover, as it could be easily foreseen in the case of a microworld, other observable artifact signs coincided with the labels of Cabri tools: <point>, <point on an object>, <construction>, <macro> (possibly the related signs <initial objects> < final objects>).

Besides these generic terms, was it possible to identify artifact signs in the names of the specific elements in play: points (<A>, …), lines <AB> , macro <Effetto1>, and the like? The following examples are drawn from the students' worksheets filled during the first session:

Excerpt 1

FE: Moving P, we realized that H is moving , whatever direction P is moving, except when P goes ⊥ (perpendicular) to the line on which H is moving, (the line) passing through B and A.

Dragging B H forms a circle, passing through P and A.

TZ: If one drags point P, H moves on the line which contains (ita. comprende) the segment AB.

Besides the predictable use of artifact signs in the description of the task solution, the analysis of their use and evolution in the subsequent activity illustrated a process of semiotic mediation centered on the artifact. It is worthwhile to identify and analyze the coexistence of artifact signs and mathematical signs and possibly the semiotic relationship that is (more or less explicitly) stated between them.

Consider the following examples, the first drawn from the first collective discussion and the second from the reports that students were asked to write.

Excerpt 2 (from the first discussion)

67. T: Well, is in your opinion, what in all this work is … the function?
68. MO: A relationship that links two points.
69. T: A relationship that links … some points … let us say which points
70. MA: That is the fact that H stays on the circle of diameter AP, on the circle of diameter …

Excerpt 3 (written by BP):

BP: The initial points are named independent variables, in fact they can be moved, individually and in our case all over the plane … H is called dependent variable and we understand easily why, it cannot move by itself, but always in function of some other moment (that is it depends on it)

These protocols show the coexistence of artifact signs and mathematical signs. We observe also how they are explicitly related. Besides the explicit definition (*the initial points <u>are named</u> independent variables)*, the explication given by the student provides a first approach to the meaning of the mathematical sign <independent variable>, referring to the free movement of the initial points and stating the first relation between motion and variation.

Motion remained a stable reference used when the nature of a particular element was to be determined. The following excerpt provides an example during the first collective discussion when a new problem arose and it became necessary to state the status of a particular point:

Excerpt 4 (from the first discussion)

289. MA: H' moves the same [italian: uguale]!
290. I: Because it is the "Effetto1" that gives it that very relationship.
291. MA: Is it possible to move H'?
292. T: Well, no, I do not think so!
293. C: What did he say?
294. T: Whether is it possible to move H'. Go on, tray?
295. GI: Hum, no!... Because now H' is obtained...
296. C: No, because H' is obtained from the first [macro/function].
297. T: H' is obtained from the first.

A group of students applied the Effetto 1 macro twice, and obtained a new point H. One of the students, Mauro, observed that "H' moves as H", but this was not enough for him to establish the status of H' as a variable. Thus Mauro asked to use the criterion in order to give an answer (291). The teacher actively participated in the construction of a shared context for validating the answer; she anticipated but also guided the empirical verification of the criterion evoked by Mauro.

To return to the analysis of the short excerpt from the written report of BP, it is possible to observe the presence and the key role of what we have called pivot signs. The sense of the definition (as BP stressed, *and we understand easily why…*) is hinged on the interpretation of the sign <independent> that draws its meaning from the context of the artifact. Thus the sign <independent> (as well <dependent>) can be classified as a pivot sign, in fact it has at the same time a reference in the mathematical context, where <independent variable> is a technical term, and a reference in the artifact context, where, according to the current interpretation in the natural language, expresses the fact that the Cabri points can freely move on the screen. This phenomenon is not isolated, take for instance the following excerpt, drawn form the same set of written reports

Excerpt 5

STE wrote: We took three points A, B, P called independent variables because they can be moved everywhere one wants and also at the same time with the button "multiple animation" … after applying the tool "Effetto1" we obtained point H called dependent variable, dependent because it moves only in function of the movement of independent variables, both individually and simultaneously.

Similar to the previous example although less explicit, STE related artifact signs (<points A, B, P>, <they move>, etc.) to mathematical signs and did it referring to the experience in Cabri. These hybrid sentences are similar to those of the abacus example in their mixing different categories of signs and witness the transition from artifact signs to mathematical signs.

Together with the previous one, this fragment is a good example of the differences that can be expected in students' articulation of signs. Nevertheless, a remarkable similarity between the two (BP and STE) formulations emerge and can be interpreted as an expression of the internalization of the temporary definitions, negotiated and shared during the collective discussion. This fragment is also a good exemplar of the key role of the pivot signs that deserve further attention.

The characteristic of pivot signs is their polysemy that means that makes them usable as a hinge fostering the passage from the context of the artifact to the mathematics context: for instance expressions like <a function of>, <depends on>, <vary>, <object>. Usually, these kinds of signs are not easily foreseen in an a priori analysis, and for their contingency character it is left to the teacher to catch the opportunity of using them, or exploiting their emergence from students' utterances.

THE TEACHER'S ROLE IN THE EVOLUTION OF SIGNS

We are going to give a short account of the analysis of teacher's action and few examples to illustrate the role played in the process of semiotic mediation, in particular in managing the process of evolution of signs. More examples can be found in (Mariotti, 2001; Cerulli, 2004), while a detailed and systematic discussion related to this particular teaching experiment can be found in (Falcade, 2006).

Data coming from the teaching experiments highlighted a recurrent action pattern, a sequence of operations belonging to different categories. Although such categories of operations, as well as the pattern that was observed, seem to have a certain generality and overcome the contingency of the experiment, further investigation is needed to validate, but also to elaborate this first result. Nevertheless, we will use this pattern to frame our exposition because we found it effective for a communication purpose. Thus, observation highlights the following recurrent sequence of categories of teacher's operations:

Ask to go back to the task
Focalize on certain aspects of the use of the artifact
Ask for a synthesis
Synthesize

In the following, we give a short description and few examples of each category.

Ask to go back to the task. Under this category, we put all the teacher's interventions asking to recall the question of the task and, consequently, the use of the artifact in that circumstance (utilization schemes). Students are asked to reconstruct (really or in their imagination) their experience with the artifact. The objective is that of reconstructing the context of the artifact and make signs emerge in relation to that experience. This kind of operations usually results in the production (or re-emergence) of artifact signs that is fundamental for starting (or re-starting) the development of new meanings and the evolution of signs.

Focalize on certain aspects of the use of the artifact. This category is complementary to the previous one and collects all the teacher's interventions focusing the attention of the students on a particular aspect of the experience (past or present). The intervention can be more or less explicit (gestures are often observed on these occasions, both for reinforcing and accomplishing the operation). The objective is that of highlighting and limiting a part of the living experience in relation to its semiotic potential. Often following a "back to the task" intervention, a focalization emphasizes certain signs, already produced and shared, selecting aspects of their meanings that are pertinent in relation to the semiotic potential and consequently to the development of the mathematical signs that constitute the education goal.

In the following, a short excerpt from the first collective discussion illustrates the interlacement of interventions belonging to the two previous typologies:

Excerpt 6

33. T: "Effetto1" condenses, hiding it, a construction that subsequently you discovered … and what does this construction do?
34. FI: It constructs a point.
35. T: It constructs a point H starting from …?
36. Chorus (C, FI, I): From three points.
37. I: but… [starting] from two circles … we used the three points to make the two circles.
38. T: Yes, because one made you discover this construction … because, what did one say to you? I mean, what did the task ask you to do?
39. I: We should say … first if I moved point A [I was asked to say] which of the points were moving and which were not moving…
40. T: Ok ... so, for instance, moving P, I see that only H moves and not only that, let us try to summarize [Italian: condensare] a bit … I see, what does it move too...?

The first part of the excerpt shows "back to the task" interventions (33, 38). This kind of operation is cyclically repeated and used by the teacher whenever she feels the need for the students to recover their experience lived in the context of the artifact. Such occurrences are only partially planned; they are mostly produced as on the spot reactions to students' behavior. The last part of the excerpt (41) shows how from recalling the task the teacher tries to select the relevant aspect. In this case she focuses on the coordination of movements, i.e., an instance of co-variation.

As said above, these two types of operations (back to the task and focalization) are complementary: one is oriented towards the context of the artifact, whereas the other is oriented outwards. It seems also that the move from one to the other type of interventions is not one way oriented: for instance, following a direction of towards the mathematical context, the action of the teacher often stops the process. It may happen that an artifact sign is temporarily given a mathematical status, in order to consolidate significant aspects.

Ask to synthesize. We put all the operations when the teacher invites the students to make explicit what they have understood in this category; usually, this is accomplished asking them to synthesize a part of the activity, for instance, the last part of the on going collective discussion. The aim is that of inducing students to make explicit personal meanings, but at the same time the particular request may also induce them to condense (synthesize) and to generalize in one sentence different experiences (although it is not automatic that it happens). In this process of generalization, the emergence of pivot signs is expected, as well the emergence of mathematical signs; synthesis produced during collective discussion, in fact, tend to involve signs previously emerged, included mathematical signs used by the teacher. These interventions are expected to foster the development of the interpersonal space within which mathematical signs might be produced and consolidated. The sharing of personal meanings through students' synthesis forms the environment within which the teacher may introduce the point of view of mathematics, and the eventually a standard terminology. The following excerpt is drawn form the first collective discussion, when the teacher asks the students to synthesize, trying to involve also who never intervened.

Excerpt 7 (from the first discussion)

211. T: Who would like to synthesize all what I have said? … but I want someone that never talked … Ma!
212. Ma: What I understood…?
213. T: Ok, go on, what did you understand?
214. Ma: I mean … there are certain things that are taken from others that are independent… that are points A, B and P; H is obtained by a construction that derives from A, B and P, thus H depends on the position …
215. T: … on the position of the three points A, B and P. Thus the function … what is it [the function] for you?
216. Ma: The function for me is … I mean it should be a construction that practically … is obtained by different means … that derive from …
217. T: From which points?
218. Ma: A, B, and P.
219. T: OK

The teacher first asks to synthesize, then mirrors the Ma's interpretation of her first request and explicitly asks, "what did you understand?" As expected, the beginning of a de-contextualization process clearly appears and the Ma's utterance contains generic terms: they cannot yet be considered completely general, but rather their functioning as pivot signs allows Ma to turn back to the particular case of moving points in the geometrical construction (artifact signs). Similarly, the expression <depends on> can be classified as a pivot sign. The teacher follows the pupils in the explication within the artifact context, but comes back to the start, asking to make explicit the personal meaning of the sign <function> and Ma restarts from a more general point of view.

This short excerpt shows how the evolution may progress: back and forth from the artifact context to the mathematical context, the pivot signs play a hinge role, connecting the two poles, but also fostering the move from one to the other. It shows also the role of the teacher: slowly, but continuously, the teacher pushes the students to leave the artifact context, selecting specific qualities from the use of the artifact to be transferred to the mathematical context.

Synthesize. The final category that was identified gathers the interventions when the teacher recovers and fixes the use of particular signs, not always and not immediately it is the case of mathematical signs. The objective of these interventions is that of explicitly ratifying the acceptance of a sign, the use and status of which are related to the mathematical context. These interventions are expected to produce stable semiotic relationships between signs and for this reason they constitute fundamental steps in the construction of a texture of semiotic chains towards the construction of mathematical knowledge. The following example illustrates a case of synthesis (159), showing once more the teacher's care in evoking the artifact context:

Excerpt 8 (from the first discussion)

159. T: Well, then what happens is that in general for a function, the points from which I start are named independent variables, because I can move them wherever I like, whilst what I obtain is named dependent variable, because it depends … on what [does it depend]?
160. I: [it depends on] the independent variables.

CONCLUSIONS

In spite of the differences in school levels, mathematical contents, and artifacts, the two examples discussed above share general features that can be related and explained referring to the common theoretical framework that also inspired the design and the implementation of the different teaching experiments. Following Hasan's description (2002), in those experiments: (1) the *mediator* was the mathematics teacher; (2) what was *mediated* was a particular mathematical content: the decimal positional system of number representation, in the case of the abacus and the notion of function, in the case of Cabri; (3) the "*mediatees*" were the pupils; (4) the *circumstances* for mediation comprised (a) an artifact (either the different types of abacus or the different Cabri tools), together with the sign systems related to and produced during its utilization and with the didactical activities organized by the teacher, in order to make the use consistent with the content to be mediated; (b) the site, i.e., the school—the mathematics classroom, as well the computer lab.

In each case, the choice of a particular artifact was determined according to the analysis of its semiotic potential. History is a rich source of suggestions and the example of the abacus does not remain an isolated one (e.g., Bartolini Bussi et al., 1999, 2005; Maschietto & Ferri, in press), but the theoretical frame elaborated above naturally fits with the use of modern artifacts too. In particular, computer-based artifacts seem to have a great potential because of their natural link with mathematics. The example of Cabri, discussed above, shows how general may be the use of an artifact as a tool of semiotic mediation.

The notion of mediation is widely used in the current mathematic education literature, but it is here elaborated in a more complex way that overcomes the basic assumption about the contribution of integrating tools in the human activity, and in mathematics in particular. Besides considering an artifact (either an abacus or a pocket calculator) a powerful prosthesis students may appropriate in order to solve given tasks, an artifact can be exploited by the teacher as a tool of semiotic mediation to develop genuine mathematical signs, detached from the use of the artifact, but maintaining with it a deep semiotic link.

Beyond and not in contrast with the objective of making the artifact become transparent or "converting tools into mathematical instruments" (Guin & Trouche, 1999), we postulate a parallel direction of development, under the guidance of the teacher, aimed to socially construct that mathematical knowledge potentially related to the use of the artifact.

The evolution of signs, corresponding to the move from personal meanings rooted in the context of the artifact to conscious mathematical meanings, is a long-term process that, according to our assumption, is neither spontaneous nor granted.

The teaching organization that we propose in this chapter is modeled by the iteration of what we called the didactical cycle. In particular, we assumed the centrality of the design of semiotic activities, such as production and interpretation of texts, and, in particular, collective mathematical discussion where the intentional action of the teacher is focused on guiding the evolution of signs.

In the two examples considered, the analysis of the semiotic potential had been realized focusing, on the one hand, the embedded knowledge, and, on the other hand, the utilization schemes that emerge when the artifact is used to solve a given task (a representation task in the case of abacus; an exploration task in the case of Cabri). This analysis allowed teachers to plan lessons and tasks, in a way that recalls the planning of semiotic chains, described by Presmeg (2006). Some fragments of the classroom implementation have been described, focusing on different components of the didactical cycle. Not all the students' reactions could be foreseen in the a priori analysis, so that the teacher was asked to adapt her intervention to unexpected contributions. Ready to give space to the evolution of students' personal meanings, the teacher had to keep in mind her goal towards mathematical meanings, without neglecting students' personal perspectives. The teacher's attitude is consistent with the theoretical construct of mathematical discussion, where different voices are given space to be elicited and developed (Bartolini Bussi, 1998). The classroom interaction processes showed very complex and long and, as said, partially unpredictable. The constitution of meaning was slow and could be revealed only over time. The time length of the process recalls what is reported by Saenz-Ludlow (2006) in the definition of cycles of interpreting games in classroom interaction. There is, however, a difference: in the Vygotskian framework assumed by us, the asymmetry of roles of teacher and students is emphasized. This marks an important difference between socio-constructivist (e.g., Cobb, 1999) and Vygotskian approaches to learning, to the extent that, in the latter, it is preferred the double expression teaching-learning (that is a better translation of the Russian word *obuchenie*). It means that the teacher is the ultimate witness and is responsible for the meaning to be constructed and appropriated by students (for a discussion of this issue see also Bartolini Bussi, 1998; Vianna & Stetsenko, 2006; Radford, chapter 18, this volume). This clearly appears in the action of focalizing and in that of synthesizing; in particular the teacher is responsible on how far the evolution is to be stretched or stopped. This may explain also why this model may be applied to secondary schools as well, where the curriculum is perceived by teachers as more intense than in elementary schools (Saenz-Ludlow, 2006, p. 215).

Research projects focused on the analysis of the process of semiotic mediation are still in progress, in particular in relation to the study of teacher's action, but the results obtained so far, and partially presented in this chapter, confirm the consistency of the theoretical frame. The analysis of the evolution of signs, as well the analysis of the action of the teacher shed new lights on the potentialities of using old as well new artifact in classroom activity, showing at the same time the complexity of the functioning of a particular artifact as a tool of semiotic mediation. The classification of teacher's operations needs to be tested in different situations so that to overcome the contingency of the particular experiment and of the particular teacher.

The construction of semiotic chains constitutes one of the goals of teacher's interventions. In the examples, reaching a mathematical definition does not only mean the production of a mathematically correct statement, but also the construction of a web of semiotic relationships supporting the construction of the corresponding mathematical concept. The construction of this web allows one to freely use artifact signs far beyond the definition of mathematical signs, without loosing the generality requested by a mathematical discourse, or to come back to such signs whenever their evocative power could be useful.

ACKNOWLEDGMENTS

This study was carried out within the research project Meanings, Conjectures, Proofs: From Basic Research in Mathematics Education to Curricular Implications funded by MIUR (PRIN 200501972).

We are deeply indebted with the teachers and the students who in past years collaborated in designing the teaching experiments and in letting us observe and analyze its realization in the classroom. The text was prepared when the two authors were in different parts of the world. Skype, the little piece of software for communicating, created in 2003 by Niklas Zennström and Janus Friis, allowed them to be in touch and to discuss everyday.

NOTES

1. There are many terms that refer to artifacts conceived for a specific use and a specific goal (e.g., tools, instruments). Because one of the aims of this chapter is to clarify some aspects related to the use of artifacts, henceforth the term will be used with the most general meaning.
2. Actually, the use of a computer, and, generally speaking, working in a computational environment, presupposes the use of writing. "Written words" appear on the screen and constitute the main communication medium. Waiting for the future evolution in the communication with a computer, now the main mode is still based on writing and reading.
3. Until now, the word artifact has been used as a generic term to mean something produced by human beings. In this section, the meaning will be specified and compared with the word instrument also to be meant in a technical sense.
4. We utilize the term *sign* in a sense deeply inspired by Pierce, and consistent with the latest claims concerning the need of enlarging the notion of semiotic system (Radford 2003; Arzarello 2006) including different and more flexible kind of signs. "A sign is in a conjoint relation to the thing denoted and to the mind. If this relation is not of a degenerate species, the sign is related to its object only in consequence of a mental association, and depends upon a habit" (Hartshorne & Weiss, 1933, p. 360). As Arzarello stresses, intentionality is a crucial character of sign which concerns people involved in their use.
5. In the English translation of Vygotsky, the word tool is used instead of artifact. Here the word tool will be used only in the expression "psychological tool" or "tool of semiotic mediation" and related meanings, and this will be done for the purpose of maintaining the consistence with Rabardel's uses of the term *artifact* and *instrument*.
6. Following Wartofsky (1979), the term *artifact* has to be meant in a broad sense, including tools like hammers, compasses, abaci, softwares, but also texts, historical sources, speech, gestures, didactical movies, exhibits of at a science center, mathematical theories, and so on). Wartofsky's (1979) claims that "What constitutes a distinctively human form of action is the creation and use of artifacts, as tools, in the production of the means of existence and in the reproduction of the species. *Primary artifacts* are those directly used in this production; *secondary artifacts* are those used in the preservation and transmission of the acquired skills or modes of action or praxis by which this production is carried out. Secondary artifacts are therefore representations of such modes of actions" (Wartofsky 1979, 200 ff.). There is also another class of artifacts (*tertiary artifacts*), "which can come to constitute a relatively autonomous 'world', in which the rules, conventions and outcomes no longer appear directly practical, or which, indeed, seem to constitute an arena of non-practical, or 'free' play or game activity. This is particularly true … when the relation to direct productive or communicative praxis is so weakened, that the formal structures of the representation are taken in their own right as primary, and are abstracted from their use in productive praxis" (Wartofsky 1979, 208 ff.)
7. According to the perspective that we take (individual or social), one can speak about the relationship between artifact and knowledge as evoked knowledge (according to an individual) or embedded knowledge (according to a historic-epistemological analysis of an artifact). In fact, for the expert signs may be evoked by the artifact, as well as the corresponding knowledge. For instance, positional notation and the polynomial notation of numbers may be evoked by an abacus; similarly, "rule and compass" geometry may be evoked by the Cabri environment. But we can say that positional notation is embedded in an abacus, as well as "rule and compass" geometry is embedded in the Cabri environment (Cerulli, 2004).
8. This process is nicely expressed in the following quotation, where the specific role of generating sign is assumed: "In other words, to arrive at the goal, the individuals relay on the use and articulation

of several artifacts and semiotic systems through which they organize their actions across space and time. […] these artefacts and varied systems of signs that individual use in social meaning making process to make apparent their intention and carry out their actions in order to attain the goal of their activities I call *means of semiotic objectification"*(Radford, 2003, p. 4).

9. The relationships between (personal) sense and meaning in the Vygotskian tradition have been elaborated in different ways by several authors (e.g., Vygotsky, 1990; Leont'ev, 1976). The original words used by Vygotsky are *smysl* (sense) and *zna enie* (meaning). The former refers to "the sum of all the psychological events aroused in our consciousness by the word. It is a dynamic, fluid, complex whole, which has several zones of unequal stability" (Vygotsky, 1990, p. 380); the latter refers to "the most stable and precise zone" (p. 380). Leont'ev (1976), on the contrary, uses the same word *zna enie* to refer to "the range of a given society's ideas, science and language" (p. 245), emphasizing the objective and historical features of it. In the following, we shall use the expression "personal meanings" when the focus is on the individual and meanings when the focus is on mankind's cultural experience.
10. The term *orchestration* is used by Trouche (2005) and by Bartolini Bussi (1998) with different meanings. In fact, the former author refers to the integration of technology into classroom practice, whilst the latter refers to the coordination of the different voices that are produced during classroom discussions (see below). The latter meaning will be used throughout this paper.
11. The iteration of cycles is consistent with the fundamental hypothesis about internalization as a long term process, as Vygotsky claims: "The transformation of an interpersonal process into an intrapersonal one is the result of a long series of developmental events."(Vygotsky, 1978, p. 57)
12. Actually, we must distinguish between the writing of reports and the writing of a notebook. Reports are written whenever the teacher asks for them and are aimed at describing what has been done and understood until then. There is no strict order in writing reports and discussions. On the contrary, writing the notebook is one of the main activity and there is a periodic revision of the notebooks.
13. In a similar way, an artifactual chain may be generated in the long term processes, when an artifacts is either substituted by or placed side by side with another artifacts. This will be the case, for instance, of the pair of Slavonic abacus and spike abacus. However, we shall not elaborate details about this chain in this paper.
14. This is the case for instance of signs as <*pallina-decina*> (see below) and <*teorema-bottone*> (Cerulli, 2004, p. 137).
15. The quoted examples are not yet presented with explicit reference to this theoretical framework. Rather they have been the germs from where the theoretical framework has been generated, deepened and enriched.
16. Here and in the following, we use DGE as Dynamic Geometry Environment and we refer to Cabri to mean any of the successive versions of Cabri-géomètre (http://www.cabri.com).
17. Consider, for instance, the shift from the empirical use of a pair of compasses to the theoretical definition of circle given by Euclid.
18. As is well known, a correspondence can be stated between construction tools and geometrical properties, although such a correspondence is not so direct and explicit as the names suggest and would require a detailed semiotic analysis to be fully explained.
19. According to the Pierce's terminology, the contribution of this icon should be preserved throughout the sequence of interpretants.
20. According to Rabardel's terminology, different instruments may be constructed in relation to these schemes which respectively evoke different mathematical meanings related to the notion of function:
    - the notion of variable, and of its variation in the domain;
    - the notion of dependent variable, and its variation in the range.

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