Educ Stud Math (2011) 77:189–206
DOI 10.1007/s10649-010-9280-3

Semiotic and theoretic control in argumentation and proof activities

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Published online: 25 November 2010

Abstract We present a model to analyze the students' activities of argumentation and proof in the graphical context of Elementary Calculus. The theoretical background is provided by the integration of Toulmin's structural description of arguments, Peirce's notions of sign, diagrammatic reasoning and abduction, and Habermas' model for rational behavior. Based on empirical qualitative analysis we identify three different kinds of semiotic actions featuring the organization of the argumentations, and related to different epistemological status of the arguments. In such semiotic actions, the students' argumentation and proof activities are managed and guided according to two intertwined modalities of control, which we call semiotic and theoretic control. The former refers to decisions concerning the selection and implementation of semiotic resources; the latter refers to decisions concerning the selection and implementation of a more or less explicit theory or parts of it. The structure of the model allows us to pinpoint a dialectical dynamics between the two.

Keywords Argumentation · Proof · Semiotic control · Theoretic control · Abduction

1 Introduction

Processes of argumentation and proving in different contexts have been widely analyzed by many scholars from a variety of perspectives. Many studies consider the structural aspects of argumentations and proofs, often using a ternary model by Toulmin (1958/2003). For example, Pedemonte (2007) uses the Toulmin's model to give account of the cognitive difficulty in the passage from abductive steps to deductions in individual processes of proof production, whereas Knipping (2008) exploits the model to reconstruct local arguments in classroom proving processes.

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On the other hand, some researchers (Dörfler, 2005; Radford, Bardini & Sabena, 2007) pointed out the relevance of the semiotic resources used and produced by the students and of the consequent activities with them. The semiotic aspects constitute in fact an important dimension of mathematics argumentation and proof. Dörfler has discussed the role of diagram in proofs, and Radford and his collaborators that of gestures and rhythms in discovering and conjecturing. According to a fundamental observation of Peirce (CP, 3.363), semiotic activities make observational and experimental aspects accessible to mathematical argumentations, showing the weakness of those positions that claim the existence of a gap between empirical and theoretical reasoning.

Along this stream, we shall show how students' argumentation and proving activities are managed and guided according to two main intertwined modalities of control, namely *semiotic* and *theoretic control*. In doing so, we shall introduce a model for describing students' semiotic processes in argumentation and proving activities, and the role of the theoretical dimension therein. The model is based on the observation of secondary students involved in an early Calculus course focused on the graphical representation. It will be described and substantiated by the qualitative analysis of students' written protocols. Before entering into details, we briefly sketch its main theoretical ingredients, namely Peirce's notions of sign, diagrammatic reasoning and abduction, Toulmin's model of argumentations, and Habermas' elaboration for rational behavior.

2 Semiotic and theoretic control: analysis tools

Peirce points out a "paradoxical feature" of mathematics, which distinguishes it from the other scientific disciplines:

> It has long been a puzzle how it could be that, on the one hand, mathematics is purely deductive in its nature, and draws its conclusions apodictically, while on the other hand, it presents as rich and apparently unending a series of surprising discoveries as any observational science. Various have been the attempts to solve the paradox by breaking down one or other of these assertions, but without success. The truth, however, appears to be that all deductive reasoning, even simple syllogism, involves an element of observation; namely, deduction consists in constructing an icon or diagram the relations of whose parts shall present a complete analogy with those of the parts of the object of reasoning, of experimenting upon this image in the imagination, and of observing the result so as to discover unnoticed and hidden relations among the parts. (Peirce, C.P., 3.363: quoted from p. 57 in Dörfler, 2005)

Mathematics happens to live in a tense dynamics between its *deductive nature* and those elements of *observation* that lead to discoveries and development. The linking ring, suggests Peirce, is constituted by *signs*[1] and by what he calls *diagrammatic reasoning*.

[1] According to Peirce (1931–1958), a *sign* is a triad composed by the sign or *representamen* (that which represents), the *object* (that which is represented), and the *interpretant*: "It [The sign] addresses somebody, that is, creates in the mind of that person an equivalent sign or perhaps a more developed sign. That sign which it creates I call the interpretant of the first sign. The sign stands for something, its object. It stands for that object, not in all respects, but in reference to a sort of idea." (C.P. 2.228)

Peirce defines diagrammatic reasoning as a three-step process:

(a) constructing a representation;
(b) experimenting with it;
(c) observing the results.

Through the diagrammatic reasoning he could overcome the epistemological paradox illustrated in the quotation above, underlying the relevance of the perceptual components in mathematical activities. As Radford (2008) points out:

> Diagrammatic thinking is a central piece in Peirce's endeavor to rescue the epistemological import of perception. It is strongly linked to a heuristic process that exhibits, via intuition (i.e., in a sensual manner), some aspects of the object under scrutiny, thereby making these aspects available for observation, in order to help us discover new conceptual relations. The epistemological potential of diagrammatic thinking rests then in making apparent some relations that have thus far remained hidden from perception or beyond the realm of our attention. p.10

In the examples, we shall discuss how diagrammatic reasoning allows a student to develop what we call *semiotic control* in a mathematical argumentation, and how it is intertwined with what we call *theoretic control.* Schoenfeld (1985) defines control in problem-solving activities as "Global decisions regarding the selection and implementation of resources and strategies". It entails actions such as: planning, monitoring, assessment, decision-making, and conscious metacognitive acts. We speak of *semiotic control* when the decisions concern mainly the selection and implementation of semiotic resources, namely when the decisions concern activities featured by the treatment of signs, according to a wide Peircean interpretation of what a sign is (from the symbols and graphs of mathematics to the drawings sketched by the students, to their gestures, etc.). On the other hand, the control is *theoretic* when the decisions concern mainly the selection and implementation of a more or less explicit theory or parts of it, and is accomplished through a more- or less-organized cluster of properties, algorithms, and possibly conceptions that students activate to elaborate an argument or a proof. For example, a semiotic control is necessary to choose a suitable semiotic representation for solving a task (e.g., an algebraic formula vs. a Cartesian graph), while a theoretic control intervenes when a students decides to use a theorem of calculus or of Euclidean geometry for supporting an argument. Of course the two modalities are often intertwined in the concrete actions of students who solve a task and are here distinguished for the sake of analysis.

A further tool from Peirce is abduction, specifically the so-called *syllogistic abduction* (C.P. 2.623[2]), according to which a *Case* is drawn from a *Rule* and a *Result.*[3] His example about beans is well known:

Rule: All the beans from this bag are white.

[2] Peirce's work is usually referred to in the form *C.P. n.m.*, which means Collected Papers; *n*, number of volume; *m*, number of paragraph.

[3] Abduction has already been widely used and discussed in mathematics education, for instance by Arzarello, Micheletti, Olivero, Paola & Robutti (1998), Hoffmann (2005), Radford (2008), and Rivera (2008).

Result: These beans are white.
Case: These beans are from this bag.

As such, an abduction is different from a deduction that would have the form: the Result is drawn from the Rule and the Case, and it is obviously different from an induction, which has the form: from a Case and many Results a Rule is drawn. Eco (1983) describes abduction as the search for a general rule from which a specific case would follow. Of course the conclusion of an abduction holds only with a certain probability (in fact Polya, 1954 called this abductive argument an *"heuristic syllogism").* The conclusion is a *plausible hypothesis*, in the sense that it is a supposition stated with some plausibility. And in this sense, Peirce (1931–1958) called also "*hypothesis"* the abduction (hence not in the sense according to which it is usually used in proofs, where a hypothesis is a statement with no necessary ground). Moreover, as pointed out by Peirce (C.P. 5.14–212), three aspects determine whether a hypothesis (abduction) is promising: it must be *explanatory*, *testable*, and *economic*. A hypothesis is an *explanation* if it accounts for the facts; its status is that of a suggestion until it is verified, which explains the need for the *testability* criterion. The motivation for the *economic* criterion is twofold: it is a response to the practical problem of having innumerable explanatory hypotheses to test, and it satisfies the need for a criterion to select the best explanation amongst the testable ones. Peirce points out that abductive reasoning is essential for every human inquiry. It is intertwined both with perception and with the general process of invention: "It [abduction] is the only logical operation which introduces any new ideas" (C.P. 5.171). In short, abduction becomes part of a *process of inquiry* in which abduction, induction, and deduction play particular roles.

A second tool for our analysis is Toulmin's model, which takes care of the *structure* of argumentations (Toulmin, 1958/2003). According to Toulmin's model, every argumentation is composed by a statement or *claim*, some *data* justifying the claim, and some inference rule (called *warrant*) allowing to link the data to the claim. This basic structure may be enriched by auxiliary elements: a *qualifier*, indicating the strength provided by the warrant; a *rebuttal*, which indicates the exceptions to the rule of the warrant; the *backing*, i.e., the grounding of the warrant. Whereas the warrant is usually mentioned in an explicit way, the backing remains often implicit. To illustrate the Toulmin's model, let us consider an example provided by Toulmin himself (see p. 92 ff. in Toulmin, 1958/2009.). "Harry is a British subject since he was born in Bermuda". "Harry is a British subject" is a claim that is supported by the data "Harry was born in Bermuda", according to the warrant "a man born in Bermuda is a British subject". This warrant relies on a certain backing, which refers to the British laws. A possible rebuttal could be, for instance, the fact that Harry has become a naturalized American.

Toulmin's analysis is aimed to "characterize what may be called 'the rational process', the procedures and categories by using which claims-in-general can be argued for and settled" (*ibid.*, p. 7). He is interested in studying the structure of arguments as a product rather than the processes through which they are generated. Following other researchers (see for instance Pedemonte, 2007; Inglis, Mejia-Ramos & Simpson, 2007; Jahnke, 2008), we shall adopt Toulmin's (complete) model to study argumentation and proof not only as products but also as processes. An innovative contribution of our work will be complementing the structural analysis that makes use of such a model with a semiotic analysis that takes into account the specificity of the mathematics argumentation. This operation is not far from the ideas of Toulmin, who argued for the strong dependence of argumentations on their specific "field", and rejected the models based on classical logics, because they claimed to be universal.

In order to account for argumentation and proving processes in the classroom, Toulmin's structural analysis has been recently integrated by Boero and his collaborators with Habermas' model of *rational behavior in discursive practices* (Boero, Douek, Morselli, & Pedemonte, 2010). They have adapted the three components of rational behavior according to Habermas (teleologic, epistemic, communicative) to the discursive practice of proving, and have identified:

1. "an *epistemic aspect*, consisting in the conscious validation of statements according to shared premises and legitimate ways of reasoning (cf. the definition of "theorem" by Mariotti et al. (1997) as the system consisting of a statement, a proof, derived according to shared inference rules from axioms and other theorems, and a reference theory);
2. a *teleological aspect*, inherent in the problem-solving character of proving, and the conscious choices to be made in order to obtain the desired product;
3. a *communicative aspect*, consisting in the conscious adhering to rules that ensure both the possibility of communicating steps of reasoning and the conformity of the products (proofs) to standards in a given mathematical culture" (see p. 188 in Boero et al., 2010).

In this model, the expert's behavior in proving processes can be described in terms of (more or less) conscious constraints upon the three components of rationality: "constraints of epistemic validity, efficiency related to the goal to achieve, and communication according to shared rules" (*ibid.*, p. 192).

As the authors point out, such constraints result in *two levels of argumentation*:

- a level (that we call *ground level*) inherent in the specific nature of the three components of rational behavior in proving;
- a *meta-level*, "inherent in the awareness of the constraints on the three components" (*ibid.*, p. 192).

According to our interpretation, the first two components and the interplay between the ground and the meta-level are particularly relevant for managing what we call the *theoretic* aspects of proving processes.

3 A model encompassing semiotic and theoretical features

We will argue that the students' argumentation activities can be analyzed according to three different kinds of the semiotic actions they produce:

1) *Interpretation of signs;*
2) *Identification of relationships between the interpreted signs, and checking with arguments;*
3) *Elaboration of arguments that explain the "why": towards a theory.*

The model is illustrated by protocols of students facing a calculus problem in the graphical register, using paper and pencil. The empirical support is meant to provide an illustration of the model, rather than being a full analysis of the students' writing and thinking during the task.

The problem was given to grade 9 students in a scientifically oriented school (5 h of mathematics per week) as an assessment task at the end of a teaching sequence on the relationships between a function graph and the graphs of its derivative and of one of its primitives (introduced as "antiderivatives", i.e., the function graphs whose derivative is the given graph). The text of the problem is the following (Fig. 1):

The drawing shows the graphs of: a function f, its derivative, one of its antiderivatives. Identify the graph of each function, and justify your answer.

During the activities in the classroom we have observed all the three types of semiotic actions in the students (see the next sections for some examples). We also realized that the teacher triggered and supported the students' semiotic actions of the three kinds. In fact his didactical aim was to make the students aware of the mathematical meaning of their semiotic activities, namely to push them towards a theoretical framework. However, it is beyond the purpose of this paper to elaborate about the teacher's role.

1) Interpretation of signs

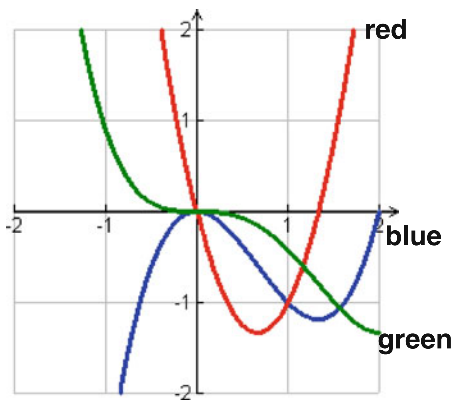
A first fundamental step in mathematical activities, included argumentation and proof, is to interpret the signs in the text of the problem, in order to identify those that are meaningful for solving the task. In the graphical problem presented above, the signs that are mostly pointed out by the observed students are: the maximum and minimum points, the zeroes, the concavity, the decreasing/increasing parts, and the points of inflection.

These signs can be considered as isolated facts, without consistent links between them. In such a case, the subject's attention is entirely devoted to the specific signs, which in our case are the main elements of the graphs. The subject's activity is therefore guided by a semiotic control, with a low theoretical concern.

2) Identification of relationships between the interpreted signs, and checking with arguments

The interpretation of signs is often accompanied and intertwined with the identification of the meaningful relationships between them. In our case, for instance a maximum of a function

Fig. 1 The graphs of a function, its derivative, and one of its antiderivatives

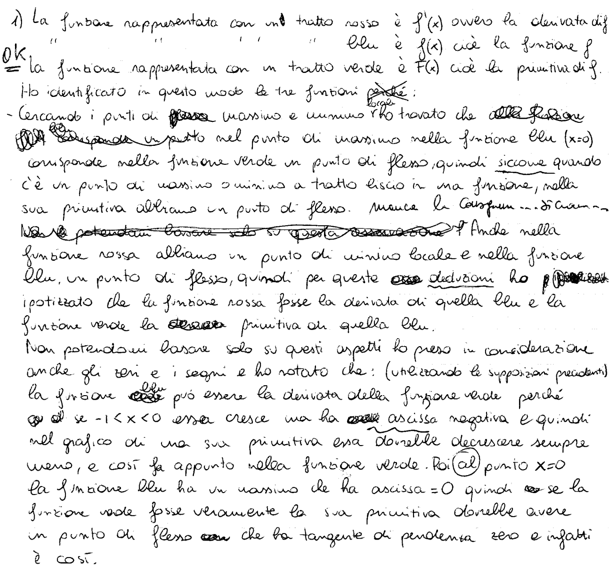


is correlated with a zero of its derivative, and with a point of inflection of an antiderivative, as shown by the protocol of Emanuela (the original is in Fig. 2):

> The function represented in red is f'(x), i.e. the derivative of f
>
> " " " blue is f(x), i.e. the function f
>
> " " " green is F(x), i.e. the primitive of f.
>
> I identified the three functions in this way:
>
> Looking for the points of local maximum and minimum I found that in the green function a point of inflection corresponds to the point of maximum of the blue function (x=0), so, because when there is a point of smooth maximum or minimum in a function, in its primitive we have a point of inflection. Moreover, in the red function we have a point of local minimum and in the blue function a point of inflection, so thanks to these deductions I hypothesized that the red function was the derivative of the blue one and the green function the primitive of the blue one.
>
> Since I could not base [my conclusions] only on these aspects I took into account also the zeroes and the signs and I noticed that: (using the previous assumptions) the blue function can be the derivative of the green one because if -1 < x < 0 it is increasing but with negative abscissa [ordinate] and so in the graph of one of its primitives it should be decreasing less and less, and it does exactly this in the green function. Then at the point x=0 the blue function has a maximum with abscissa =0, so if the green function were truly its primitive it should have a point of inflection with tangent with zero slope and in fact it is so.

With suitable arguments, Emanuela reports how she has observed and interpreted some signs and made connections between them. For example: "Looking for the points of local maximum and minimum I found that in the green function a point of inflection corresponds

Fig. 2 Emanuela's protocol


1) La funzione rappresentata con un tratto rosso è f'(x) ovvero la derivata di f
OK " " " " blu è f(x) cioè la funzione f
la funzione rappresentata con un tratto verde è F(x) cioè la primitiva di f.
Ho identificato in questo modo le tre funzioni:
- Cercando i punti di massimo e minimo locale ho trovato che nel punto di massimo nella funzione blu (x=0) corrisponde nella funzione verde un punto di flesso, quindi siccome quando c'è un punto di massimo o minimo a tratto liscio in una funzione, nella sua primitiva abbiamo un punto di flesso. manca la conseguenza ... dicam...
Anche nella funzione rossa abbiamo un punto di minimo locale e nella funzione blu, un punto di flesso, quindi per queste deduzioni ho ipotizzato che la funzione rossa fosse la derivata di quella blu e la funzione verde la primitiva di quella blu.
Non potendomi basare solo su questi aspetti ho preso in considerazione anche gli zeri e i segni e ho notato che: (utilizzando le supposizioni precedenti) la funzione blu può essere la derivata della funzione verde perché se -1 < x < 0 essa cresce ma ha ascissa negativa e quindi nel grafico di una sua primitiva essa dovrebbe decrescere sempre meno, e così fa appunto nella funzione verde. Poi al punto x=0 la funzione blu ha un massimo che ha ascissa = 0 quindi se la funzione verde fosse veramente la sua primitiva dovrebbe avere un punto di flesso che ha tangente di pendenza zero e infatti è così.

to the point of maximum of the blue function (x=0)". We have highlighted with square box the references to observed signs, and with circles some keywords referring to the organization of the argumentation. The phrases "I found", "We have", "and so it does indeed", "it is so" refer to the interpretation of the graphs, whereas "so", "deductions", "hypothesized", and "because" organize the discourse from a theoretical and logical point of view. These two aspects appear not as relegated into two separated moments but indeed as intertwined throughout the text.

In most cases, the students not only identify some relationships between the main elements of the graphs, but make also conjectures and check them with suitable arguments, like Emanuela. The checking is often redundant, i.e., the students perform more checks than what is necessary from a theoretical point of view. To pay attention to use the minimal number of passages in an argumentation is in fact one of the most sophisticated competences in mathematics, which is a step forward, towards more theoretical aspects. We focus here in detail on two phenomena that we have identified in our analysis: the *local arguments* (analyzed with Toulmin's model) and the *logic of not* (analyzed in Peirce's frame).

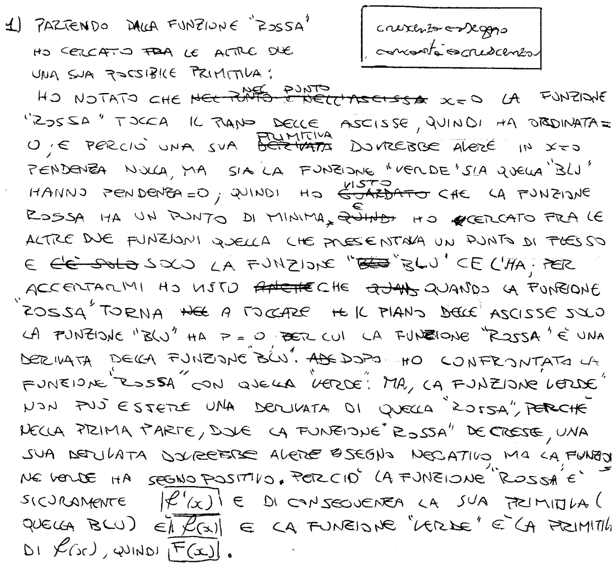
3.1 Local arguments

To organize the perceived information in their discourse, the students often made use of "local arguments", i.e., arguments that focus on specific subsets of the elements of the problem. For instance, in the structure of Emanuela's argumentation (Fig. 3) we can identify four local arguments:

1. "Looking for…point of inflection"
2. "Moreover, in the red function…the blue one"
3. "Since I could not…in the green function"
4. "Then at the point…it is so".

In the first and the second arguments she explains how she identified the functions; in her words, how she "hypothesized that the red function was the derivative of the blue one

Fig. 3 Simone's protocol



1) PARTENDO DALLA FUNZIONE "ROSSA" HO CERCATO ~~FRA~~ LE ALTRE DUE UNA SUA POSSIBILE PRIMITIVA: HO NOTATO CHE ~~NEL PUNTO E NELL'ASCISSA~~ NEL PUNTO $x=0$ LA FUNZIONE "ROSSA" TOCCA IL PIANO DELLE ASCISSE, QUINDI HA ORDINATA = 0; E PERCIÒ UNA SUA ~~DERIVATA~~ PRIMITIVA DOVREBBE AVERE IN $x=0$ PENDENZA NULLA, MA SIA LA FUNZIONE "VERDE" SIA QUELLA "BLU" HANNO PENDENZA = 0; QUINDI HO ~~GUARDATO~~ VISTO CHE LA FUNZIONE ROSSA HA UN PUNTO DI MINIMA, E ~~QUINDI~~ HO CERCATO FRA LE ALTRE DUE FUNZIONI QUELLA CHE PRESENTAVA UN PUNTO DI FLESSO E ~~C'È SOLO~~ SOLO LA FUNZIONE "~~BLU~~ BLU" CE L'HA; PER ACCERTARMI HO VISTO ~~ANCHE~~ CHE ~~QUAN~~ QUANDO LA FUNZIONE "ROSSA" TORNA ~~NEL~~ A TOCCARE ~~IL~~ IL PIANO DELLE ASCISSE SOLO LA FUNZIONE "BLU" HA $P=0$ ~~PER~~ CUI LA FUNZIONE "ROSSA" È UNA DERIVATA DELLA FUNZIONE "BLU". ~~ADE~~ DOPO HO CONFRONTATO LA FUNZIONE "ROSSA" CON QUELLA "VERDE": MA, LA FUNZIONE "VERDE" NON PUÒ ESSERE UNA DERIVATA DI QUELLA "ROSSA", PERCHÉ NELLA PRIMA PARTE, DOVE LA FUNZIONE "ROSSA" DECRESCE, UNA SUA DERIVATA DOVREBBE AVERE SEGNO NEGATIVO MA LA FUNZIONE VERDE HA SEGNO POSITIVO. PERCIÒ LA FUNZIONE "ROSSA" È SICURAMENTE $\boxed{f'(x)}$ E DI CONSEGUENZA LA SUA PRIMITIVA (QUELLA BLU) È $\boxed{f(x)}$ E LA FUNZIONE "VERDE" È LA PRIMITIVA DI $f(x)$, QUINDI $\boxed{F(x)}$.

and the green function the primitive of the blue one”. She links the local maximum and minimum of a function with the points of inflection of its primitive. Then, in the third and fourth local arguments she describes how, feeling that she “could not base only on these aspects”, she took into account the zeroes and the signs to check her conjectures. She performed local checks on the point with $x=0$, and on the interval $-1<\times<0$. All the elements that Emanuela uses to create and check her conjecture have been previously discussed in the classroom with the teacher. Now Emanuela organizes them suitably in order to solve the task.

In line with other researchers (e.g., see Knipping, 2008) we use Toulmin's model to focus more closely on Emanuela's local arguments. For reason of space, we will limit our analysis to the first and third one.

First local argument (identification):

> Claim: the function represented in blue is $f(x)$; green is its primitive.
> Data: to the point of maximum of the blue function ($x=0$) corresponds in the green function a point of inflection.
> Warrant: to a point of maximum of a function corresponds a point of inflection in its primitive (“when there is a point of smooth maximum or minimum in a function, in its primitive we have a point of inflection”).

Third local argument (checking):

> Claim: the blue function can be the derivative of the green.
> Data (implicit): in $-1<\times<0$ the blue function decreases and is negative.
> Warrant: if in an interval a function is negative and increasing, its primitive in that interval is decreasing less and less.

The warrant is explicitly mentioned in both cases, though in the former case (for the identification of the functions), in a factual form. What is most interesting from our point of view is that in the latter case the data are constituted by the perceived signs and are left implicit. Rephrasing according to our model, *the semiotic actions of the second kind are constituted by local arguments in which the considered signs enter implicitly as data*. The focus of attention of the student in her semiotic actions oscillates between the signs and the organization of the discourse. This feature is even more clear in the following case, where we discuss a peculiar way of reasoning that we call the “logic of not”.

3.2 The “logic of not”

The “logic of not” is an interesting epistemological and cognitive aspect of argumentation. A paradigmatic example is given by Simone. The strategy of Simone is similar to the one of a chemist, who in the laboratory has to detect the nature of some substance. He knows that the substance must belong to one of three different categories (a, b, c) and uses suitable reagents to accomplish his task. For example, he knows that if a substance reacts in a certain way to a certain reagent it may be of type a or b but *not* c, and so on. In such practices, abductive processes are usually used: if, as a *Rule*, the substance S makes blue the reagent r and if the *Result* of the experiment shows that the unknown substance X makes blue the reagent r, one gets that $X=S$ (*Case* of the abduction[4]).

[4] Observe that an abduction is an argument in the sense of Toulmin (see Pedemonte, 2007): the Case is the Claim (possibly with a Qualifier), the Result is the Data and the Rule is the Warrant.

In the protocol of Simone, we find arguments that are of abductive nature and reveal teleological and sometimes epistemic aspects of rational behavior. Figure 3 shows the original protocol in Italian: here we report its translation in English, parceled and numbered for the sake of analysis:

0: Starting from the "red" function
1: I looked for a possible primitive among the other two:
2: I noticed that in the point $x=0$ the "red" function touches the plane of abscissas, so it has ordinate=0;
3: and therefore any of its primitives should have in $x=0$ null slope,
4: but both the "green" function and the "blue" function have slope=0;
5: so I saw that the red function has a point of minimum,
6: and I looked among the other two functions for the one with a point of inflection
7: and only the "blue" function has it;
8: to check [this] I saw that
9: when the "red" function comes to touch the plane of abscissas again,
10: only the "blue" function has s[slope]=0,
11: therefore the "red" function is a derivative of the "blue" function.

Simone first describes two phases of his inquiry (part A, lines 1–4; part B, lines 5–8). In both parts, he describes what we could call an abductive attitude, namely how he has been looking for Results that allow him to state a Case because of a Rule. More precisely, Simone starts with f=red function (line 0), probably because it is the simplest graph, and wonders whether he can apply an abductive argument to the blue or to the green function. In both parts, the Rule is: "any primitive of f has property Q"; the Result is: "a specific function h has property Q"; the Case is "h is a primitive of f".

In part A, the Rule (Warrant according to Toulmin) is in line 3, the Result (Data) is in lines 2 and 4, while the Case (claim) is contained implicitly in line 5, which states that the first inquiry has not been successful (it was not economic) and starts a new inquiry. In Part B we have a new abductive process with a new Rule (implicitly contained in lines 5 and 6), a new Result (line 7), a judgment about the validity of the abduction (line 7) and a Case, which is not made explicit, but is implicitly stated in line 7.

Afterwards Simone checks his hypothesis, as he describes in the successive part of the excerpt (part C, lines 8–11): he is successful with a fresh abductive argument. Recalling the metaphor with the chemist experiment, Simone has been able first to find a reagent that discriminates between the substances he is analyzing, and then to confirm his hypothesis with a further discriminating experiment; that is, he has been able first to produce an hypothesis through an abduction, and then to corroborate the hypothesis through a further abduction. The experiments of the chemists are here the practices with the graphs of functions. As we will sketch later, such practices with graphs are example of diagrammatic reasoning, according to the definition of Peirce: "by experimenting upon the diagram and observing the results thereof, it is possible to discover unnoticed and hidden relations among the parts" (CP 3.363, 1885: quoted from p. 48 in Hoffmann, 2005).

Hence in parts A, B, and C Simone has produced and checked the Case of line 11: line 11 is the claim in a complex argument whose data are the Results of episodes B and C and whose warrants are the Rules of the same episodes.

Using the Habermas model, we see that some of Simone's sentences are *teleological* and at the *meta-level*: they address the successive actions of the students and his control of what is happening. The teleological component at the meta-level intertwines with the epistemological component at the ground level in a deep unity between the two. The excerpt above can be coded as follow (a ⇒ b means that the sentence a is at the meta-level and controls the sentence b at the ground level):

0, 1, 3 ⇒ 2, 4, 5
6, 8 ⇒ 7, 9, 10, 11

It must be observed that the sentences at the ground level have an epistemological component, e.g., they are logically linked to each other, because of the influence of those at the meta-level. This allows Simone to produce a proof by contrapositive. In fact now a new story comes in, marking a transition to a new epistemological status of Simone's statements. We have called such a fresh status of statements “the logic of not”. Let us explain it through what is written in part D (lines 12–15):

12: Then I compared the “red” with the “green” function:
13: but, the “green” function cannot be a derivative of the “red” one,
14: a: because in the first part,
b: when the “red” function is decreasing,
c: its derivative should have a negative sign,
15: but the “green” function has a positive sign.

Here, the structure of the sentence is more complex than before. Simone's thinking conforms to a possible abductive argument of the following form:

(1) Rule: “any derivative of a decreasing function is negative” (lines 14)
(2) Result: “the function h is negative”;
(3) Case: “the function h is the derivative of f”
(ARG. 1)

But the argument is a refutation of this possible abduction (line 13); namely it has the following form:

(1) “any derivative of a decreasing function is negative” (lines 14)
(4) “the “green” function has a positive sign” (line 15)
(5) “the “green” function cannot be a derivative of an increasing function” (line 13).
(ARG. 2)

In terms of the structure of the possible abduction ARG 1, it has the form: (1) and not (2); hence not (3). It is crucial here to observe that also the refutation of the usual Deduction (Rule, Case; hence Result) has the same structure, because the converse of the implication “A implies B” is equivalent to “not B implies not A”; namely: *(1) and not (2); hence not (3)* is the same as *(1) and (4); hence (5).* In other words, the refutation of an argument drawn through an abduction coincides with the refutation of an argument drawn through a deduction: while abductions and deductions are different, their refutations are exactly the same argument. So Simone is able to produce this form of deductive argument in a very natural way, namely within an abductive modality. This is remarkable from an epistemological point of view: whereas the abductive approach appears very natural for students in conjecturing phases (Arzarello et al., 1998), there is often a break with the deductive approach of the proving phase (Pedemonte, 2007). In fact, the transition from an abductive to a deductive modality requires a sort of "somersault", namely an inversion in the way things are seen and structured in the argument (the Case and the Result functions in the argument are exchanged) and this may be a cognitive load for the students. This inversion is not present in case of refutation of an abduction; insofar as it coincides with the refutation of a deduction. Of course, a greater cognitive load is required to manage the refutation of an abduction compared with that required to develop a simple direct abduction. But the coincidence between abduction and deduction in case they are refuted allows avoiding the somersault, as appears clearly in the protocol of Simone.

3) Elaboration of arguments that explain the "why": towards a theory

The third kind of semiotic actions is characterized by a strong emphasis on the mathematical theoretical aspects. In the students' protocols, we find arguments that explain the "why" of previous propositions with reference to properties and theorems of Elementary Calculus.[5] An example is given by the protocol of Francesca, shown in Fig. 4.

From the outset, Francesca's text differs from the previous ones in its appearance. In fact, it presents a very neat structure, and each part is clearly marked with formatting markers. Such a care for the organization of the text is an index that the girl has a strong communicative intentionality. As we shall see, the *communicative intentionality* intertwines with a *theoretic control* over the activity. Going into details with Toulmin's model, we can identify (see Fig. 5) three claims, highlighted in capital letters and in schematic way in the first three lines, two sets of local arguments introduced by two formatting dots, and finally a backing, i.e., the table.

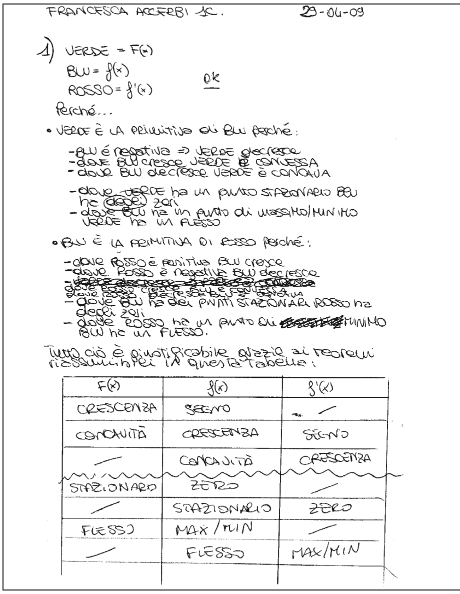
It is easy to check that the local arguments have many more warrants than necessary, constituted by one-step short statements, and juxtaposed one after the other.[6] But a most interesting element follows them: the table (Fig. 6), which we interpret as a *backing* for the argumentation. In Toulmin's model, the backing reports the grounding of the warrant. For instance, in the case of a legal argument, the backing can be the Constitution Chart of the country. Usually the backing is left implicit, and it is only mentioned to give special emphasis to the argumentation, or for specific purposes. In Francesca's case, the table is provided to intentionally point to a set of "general rules" upholding her argumentation from a very general point of view. In fact, she introduces the table as a collection of "theorems" that justify the previous (local) arguments ("All this is justifiable thanks to the theorems summed up in this table:"). The table is general, since it can work as a "theoretical kit" to solve the whole sets of problems of the same kind. It marks a theoretical control framing both the interpreted signs and the identified relationships between them.

We find similar tables, or parts of it, also in other protocols in the same classroom.[7] For instance, the reader can go back to Simone's protocol to see traces of such a structure (the top-right corner, reported in Fig. 7).

[5] In the classroom, Elementary Calculus is a theory under construction.

[6] It is a case of *parallel argumentation*, i.e., different arguments are used to support the same conclusion, as described in p. 434 in Knipping (2008).

[7] Recalling that the task we are analyzing is the final one of a teaching sequence, the genesis of this kind of table as a backing for the argumentations can be traced in the previous classroom discussion. In fact, with the guidance of the teacher a table similar to Figure 6 had been collectively built at the blackboard. It worked as a synthesis of the conclusions reached in the various activities on the relationships between the functions graphs and their derivatives and primitives. The mediation of the teacher has certainly played an important role. Though it could be very interesting *per se*, for the purposes of the paper we do not provide an analysis of the activity from a didactic point of view. We give only some informative elements to help the reader to situate our protocol analysis.



GREEN = F(x)
BLUE = f(x)
RED = f′(x)
Because…

- GREEN IS THE PRIMITIVE of BLUE because:
  - BLUE is negative → GREEN is decreasing
  - Where BLU is increasing GREEN is CONVEX
  - Where BLUE is decreasing GREEN is CONCAVE
  - Where GREEN has a stationary point BLUE has zeroes
  - Where BLUE has a MAXIMUM/MINIMUM point GREEN has a point of inflection
- BLUE IS THE PRIMITIVE of RED because:
  - Where RED is positive BLUE is increasing
  - Where RED is negative BLUE is decreasing
  - Where RED is increasing BLUE is CONVEX
  - Where RED is decreasing BLUE is CONCAVE
  - Where BLUE has stationary points RED has zeroes
  - Where RED has a MAXIMUM/MINIMUM point BLUE has a point of INFLECTION.

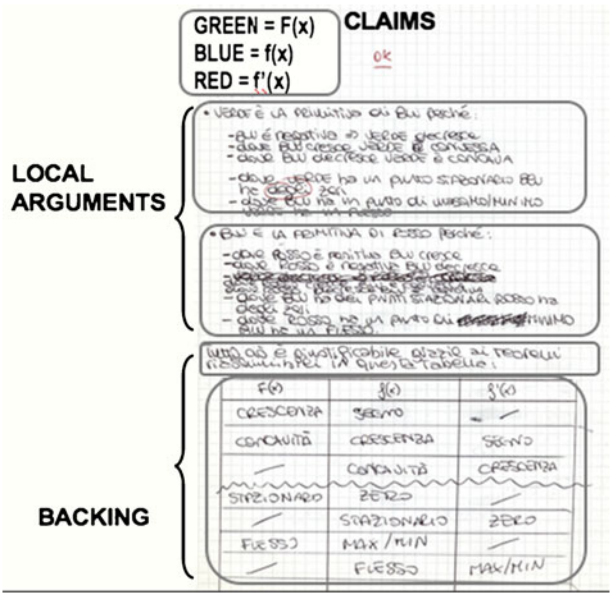
All this is justifiable thanks to the theorems summed up in this table:

[The translation of the table is in Figure 6]

Fig. 4 Francesca's protocol and its translation into English

The two lines with a linking arrow correspond to the first two rows of Francesca's table. However, in Simone's protocol the information statements are fewer than in Francesca's, and it is also much more poorly organized: there is not explicit reference to the functions (F, f, f′), and the identified correspondences are not referred to in an explicit way in the overall argumentation (see Fig. 3).

Fig. 5 The structure of Francesca's argumentation



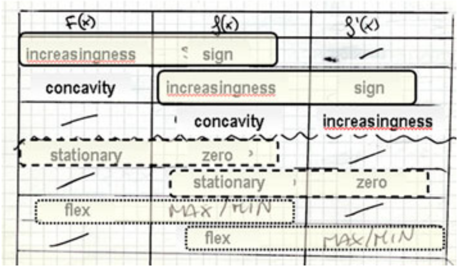


Fig. 6 Francesca's table (translated in English) and its formal analogies

Semiotic actions of type 3 differ from those of type 2 in the way the argumentation is organized. When the students are more concerned with the mathematical theory (in our case, some theorems of Elementary Calculus), they are looking for *minimal information* and organize it in a *schematic way*. In the case of Francesca, we can still find redundant elements (see the juxtaposed local arguments), but they appear purposely organized in a very structured way: observe how she makes use of dots and spaces to organize the text in distinct paragraphs. The protocols show formatting markers such as capital letters, italics, inverted commas, and underlined text, to highlight the key elements in the discourse. In this way, the organization of the argumentation is provided not only by specific words but also by *how* these words are presented. Using Peirce's semiotic theory, we can say that the argumentation is carried out both through the reference to the *objects* of the signs, and by means of their *representamen*.

Another difference from the previous cases (Emanuela and Simone) regards the content: Francesca's protocol is de-timed. All the relationships are described *sub specie aeternitatis*, namely as mathematical relationships, independent of the processes that have produced them. We cannot find traces of abductive processes: Francesca's sentences are "mathematical in structure", according to the quotation of Toulmin. All the sentences may possibly have been produced through abductions, but their epistemological status is now of mathematical sentences that are logical consequences of the theorems "summed up in the table".

4 Conclusion

We have introduced a model for the analysis of students' activities of argumentation and proof while solving graphical problems in Elementary Calculus. The model uses Toulmin's description of arguments, Peirce's notions of sign, diagrammatic reasoning and abduction, and Habermas' frame for rational behavior. It distinguishes three kinds of semiotic actions produced by the students in their argumentation and proving activities:

1) Signs are interpreted.
2) Relationships between signs are examined and checked with redundant local arguments; (economic, explanatory, and testable) hypothesis are detected and made explicit by means of abductions.

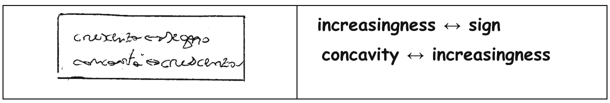


Fig. 7 Simone's correspondences

3) Argumentations explaining the relationships between signs within a mathematical theory are looked for; explanations are given through a deductive modality.

Table 1 shows the relationships between the semiotic actions and the analysis tools used in the theoretical framework.

The three kinds of semiotic actions pointed out in the model are not to be intended in a hierarchical way. Nor does each of them rule out the others. As it is well known to experts (e.g., see Thurston, 1994), discovering and proving new theorems require the intertwining of all of them.

However, each kind of semiotic action requires focusing the attention on different elements. In the first kind, when interpreting the signs that are relevant for a given task (e.g., the graphs and their features), attention is mainly concentrated on the signs themselves. Perception has a relevant role in this process. We intend perception as a complex cognitive activity, in which the subject's knowledge and the cultural dimension of the task play important roles:

> "Husserl (1997), the father of phenomenology, wrote, "[p]erception is not some empty "having" of perceived things, but rather a flowing lived experience" (p. 84). Drawing from Husserl's work, Merleau-Ponty (1945) suggested that what makes an object become an object of perception is our attitude toward it, our way of attending it, or the questions that we are trying to answer (p. 325)" (Radford et al., 2007, p. 510).

In the second kind of semiotic actions, the attention oscillates between the relationships between the interpreted signs, and their organization in coherent arguments. Usually, many arguments taking into account subsets of the perceived signs ("local arguments") are juxtaposed in a redundant manner. In Toulmin's model, the interpreted signs (actions of type 1) become the implicit Data of the arguments. From a cognitive point of view, abduction has an important role at this point, as well as the teleological component (according to Habermas) of the rational behavior in proving. Finally, in the third kind the relationships between signs are explained within a mathematical theory, with arguments that have deductive and formal features. While producing semiotic actions of the third kind, the students appear to be conscious of the logical and theoretical relationships between the various arguments (epistemic aspect), as well as of the communicative constraints required for a text to be a mathematical proof. In our protocols they often speak in terms of "theorems", "axioms", and point out formal analogies: see the many formal analogies

Table 1 The threefold model of semiotic actions

Semiotic actions	Kind of argumentation (Peirce's frame)	Structural elements (Toulmin's model)	Rational behavior (Habermas' model)
Interpreting the relevant signs	Phenomenological descriptions	Claims	–
Identifying and describing the relationships between the interpreted signs	Abductions; "logic of not"	Redundant local arguments	Epistemic and teleological aspects
Producing a text that justifies the relationships between signs within a mathematical theory	Deductions	Explicit warrants and backings	Epistemic aspects (ground level) and communicative aspects (meta-level)

contained both in Francesca's table (we highlight them Fig. 6), and in the preceding text containing the local arguments.

From our direct classroom observation and protocol analysis, the students' argumentation and proving processes appear managed and guided according to two main intertwined modalities of control, which we term *semiotic* and *theoretic control*. As we pointed out above, the former concerns decisions about the selection and treatments of semiotic resources, the latter the selection and implementation of a theory or parts of it, through a more or less organized cluster of properties, algorithms and conceptions that students activate to elaborate an argument or a proof. We have noticed an evolution from a phase where the attention is mainly on the given signs, towards a phase where the logical–theoretical organization of the argument becomes the center of the activities and evolves from abductive to deductive and more formal structures. In the model, such an evolution implies a passage from actions of type 1 to actions of type 2 and then 3, and a *shift of control* by the student, i.e., passing *from actions guided by semiotic control to actions guided by theoretical control*.

This shift has epistemological, cognitive, and didactic implications.

From an epistemological point of view, passing from type 1- to type 3-semiotic actions means an evolution *from the truth because of the data to the truth because of theoretical reasons*. It is exactly this distinction that makes the difference between what Toulmin calls a "substantial argument" and an "analytical argument", which is a mathematical proof (see pp. 114 ff. in Toulmin, 1958/2003). The crucial point is the reference in the backing to a theory (in the case of mathematical proofs) or to substantial facts (in arguments that are not mathematical proofs, for example in the legal domain). An example can clarify this distinction between analytical and substantial arguments. Suppose you ask if the following argument is valid:

Since

(1) No right-angle triangle is equilateral
(2) Some isosceles triangles are equilateral

it follows that

(3) Some right-angle triangles are not isosceles.

Many people answer that the argument is valid (since all the three sentences are true). Now ask them if the following argument is valid:

Since

(1) No dog is a ruminant
(2) Some quadrupeds are ruminant

it follows that

(3) Some dogs are not quadrupeds

The answer generally is that the argument is not correct (since the last sentence is false). But the two arguments are the same, in the sense that they have the same form, namely:

Since

(1) No S is M
(2) Some P is M

it follows that

(3) Some S is not P

The first answer is given according to an idea of truth because of the data, whereas understanding why it is not so (namely that the form of the argument must be considered, and not its specific instantiations) requires the idea of truth because of theoretical reasons.

From a cognitive perspective, our model points out also some key features of the processes that mark the transition between the different semiotic actions: the use of local arguments and abductions, and the “logic of not”. By means of the “logic of not” the students appear able to perform refutations that in principle are very difficult from a cognitive point of view, and this is due to the fact that, as we have seen, refutations of abductions and of deductions coincide. An interesting research problem for future studies is to contrast the cases of *productions* of abductions and deductions with those of their *refutations*.

From a didactical point of view, in the observed classroom the transition to theoretical aspects of argumentation and proof was guided by the teacher's didactical design of the activities and his interventions. The acquisition of a theoretical perspective is in fact a long-term didactical goal in the secondary school, and we think that the students need the support of the teacher in order to achieve it. Our research plans include deepening the investigation in this direction.

Acknowledgment This study is jointly funded by the MIUR and the Università di Torino (PRIN 2007B2M4EK).

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