A PHENOMENAL APPROACH TO MATHEMATICS

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INTRODUCTION

Mathematics is often presented as a collection of disparate topics, each with its associated techniques, typical problems, concepts and language, images and classic student errors (Griffin & Gates ref), Mason & Johnston-Wilder 2004). Each topic is the distillation of a method which someone found for solving a class of problems. As with arithmetic, where the four operations are abstracted from contexts and only later are contexts reintroduced, methods constituting topics are often isolated and abstracted from any originating context. Only towards the end of the study of the topic are 'applications' introduced. Since learners are tested on the use of techniques on routine questions, teachers are tempted to follow textbooks in demonstrating techniques using worked examples, and then inviting learners to follow the template, or as Gillings (1961) reports from Egyptian papyry of 200 BCE, to 'do thou likewise'.

In order to provoke learners into taking initiative, into engaging fully with mathematical ideas and mathematical thinking, it is necessary to construct pedagogic tasks which call upon learners to make use of their undoubted powers of making sense. Those powers include

imagining & expressing what is imagined;

particularising, specialising, & generalising (Polya 1962, Mason *et al* 1984);

conjecturing & convincing yourself and others;

organising & characterising;

focusing and de-focusing.

These powers are involved in all of human sense making, but form the core of mathematical sense making. For descriptions and examples of these and similar powers, see Dewey (1938) Cuoco *et al* (1996), Mason & Johnston-Wilder (2004).

Many authors have observed that what activates human sense making is disturbance, experienced as surprise, as puzzlement or perplexity (Dewey *op cit*, Heidegger 1962), as recognition that conscious or unconscious expectation has been broken, or as dissonance, whether cognitive (Festinger 1957), affective, or enactive. Paul Halmos makes use of this by always beginning with a question:

> Let me emphasize one thing ... the way to begin all teaching is with a question. I try to remember that precept every time I begin to teach a course, and I try even to remember it every time I stand up to give a lecture... [Halmos 1994 p852]

The claim put forward here is that mathematical thinking can be initiated through experiencing some phenomena which trigger questions. These phenomena may be drawn from the material, imaginal, symbolic, and social worlds which human beings inhabit (corresponding to Bruner's three modes of representation: enactive, iconic and symbolic with the addition of the social). Once initiated, that thinking needs to be fostered and sustained through the use of mathematically conducive ways of working, into which learners are socially enculturated through being in the presence of a teacher who not only manifests relevant behaviour, but is aware of that behaviour and uses techniques to promote that behaviour in learners. Ways of working has been variously described: *How To Solve It* (Polya 1957, 1965), *scientific debate* (Legrand 1993 see also Mason 2001), a *conjecturing atmosphere* (Mason *et al* 1982), *ways of working* (Tahta & Brookes 1966) and so on. And as Polya pointed out, most vital for actual learning as distinct from mere participation is the phase of *looking back*, of making sense of what has been done, of entering as fully and vividly as possible key moments of the work (Tripp 1993) in order to strengthen links between the state of being stuck and strategies for getting unstuck (Mason *et al op cit*, Mason 2002).

In this short paper I will only be able to offer a number of examples of phenomena which invoke learners' mathematical sense making and which illustrate some of the possibilities. In my presentation I shall use different examples. The underlying conjecture is that associated with every topic there is a surprise or disturbance, as well as a frisson of pleasure at being able to encompass a whole class of problems in a single method or way of thinking. The teacher's job at the beginning of a topic is to re-experience, to re-enter that surprise in and for themselves (Moshovits-Hadar 1988), in order to create conditions in which learners too can experience that initiating energy.

SOME SAMPLE PHENOMENA

ROLLING CUP

Phenomenon A plastic cup rolls about on the floor.

Questions What path will the cup follow, and what dimensions of the cup do you need to know in order to predict details of the path? Could two cups of different shapes roll on the same path, in some sense?

Possible developments Predicting the path shape is based on experience of the world, though it is not so easy to be precise about why it must have that shape. Predicting details of the curve makes use of properties of circles and the use of ratios. There are opportunities to work on seeking and expressing relationships and on generalising rather than simply dealing with a particular cup.

Elaboration

A simple phenomenon, observed by many people, many times, nevertheless has the potential for offering learners experience of mathematising. The real issue with techniques is not so much to learn to apply them, which is a pedagogic issue, as to recognise that a particular technique could be of use: knowing-to use a technique is psychologically quite different from knowing-how to use it, or even, given the technique, knowing-when to use it. Knowing-to requires something in a situation to bring a relevant technique to mind.

Drawing a diagram is part of the movement from material or experienced phenomenon to the imagistic world of diagrams and icons, on the way, usually, to the symbolic world of formulae and algebraic or other expressions. In order to draw a relevant diagram certain features have to be stressed (e.g. the cross-section where the cup touches the floor) and others consequently ignored (the colour and material of which it is made). Furthermore the circularity of the rim and bottom are integrated into the awareness that what matters is the ratio of the top and bottom rims and the vertical height between them. The diagram then needs to be extended to show where the centre of the rotation will be as the cup rolls about, permitting some symbolic calculations in order to determine the radius of the rotation circle (whether measured at the top or bottom rims).

MOVING FINGER

Phenomenon A metre rule or other similar stick rests on one finger on each hand. The two fingers move towards each other smoothly and uniformly. Watch the stick's movement.

Questions Why does that happen?

Possible developments Clearly friction is relevant, and centre of gravity is likely to come into any explanation. There are opportunities to imagine what will happen before it starts, then to try reconcile prediction with observation: the role of mathematical modelling. Is there a two dimensional version?

SAUSAGES & CHOCOLATE BARS

Phenomenon A number of identical sausages are to be divided fairly among a number of people.

A number of chocolate bars are to be divided fairly among a number of people.

Questions What is the least number of cuts?

Possible developments Modelling assumptions that ends of sausages are no different from middle bits, perhaps; Opportunity to compare methods, to widen appreciation of ways of depicting fractions; contact with extremal problems.

MIDPOINT

Phenomenon Imagine a parabola, and chord between two points on the parabola. Pay attention to the midpoint of the chord.

Questions What are possible positions of the midpoint as both ends of the chord move freely on the parabola.

Possible developments What about for a cubic? Suddenly all sorts of preconceptions about curves surface. Related questions include assigning to each point of the plane the number of tangents to a given cubic through that point, and looking for the boundaries of regions of points with the same value. This can be extended to quadratics and indeed to any differentiable function.

PARABOLA

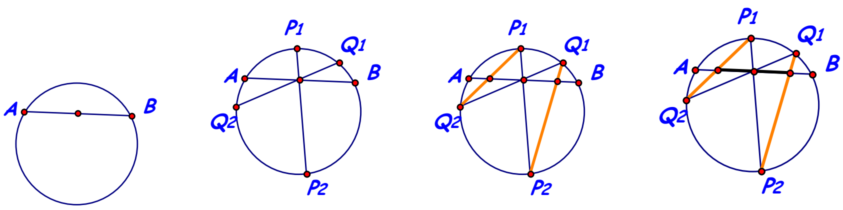
Phenomenon Imagine four points on a parabola. Draw the straight line through each pair of points. The lines meet in pairs in seven points, four of which are the original ones. Draw the circle through the remaining three points and note the position of the centre. Drag or otherwise move the four points about on the parabola.

Questions What is the locus of the centre of the circle?

Possible developments Use of coordinate geometry. What is special about the parabola?

CIRCLES

Phenomenon



Questions Is the point which is the midpoint of AB, of P1Q1, and P2Q2 always the midpoint of the segment cut off by P1Q2 and P2Q1 on AB?

Possible developments Looking for relationships, reasoning from definitions

Looking for generalisations and-or extensions

NUMBER PATTERNS

Phenomenon $1 = 1^3$, $3 + 5 = 2^3$, $7 + 9 + 11 = 3^3$, $13 + 15 + 17 + 19 = 4^3$, …

Questions Does this continue? What is the 'this' that continues? Is it always true?

Possible developments Are there some variations. Opportunities to imagine, to detect relationships and express these as general statements; opportunity to use symbols to express relationships.

(-1) X (-1) = 1

Phenomenon From a spreadsheet or other animation, some entries appear in a grid as shown below

				1 x 3 = 3		
			0 x 2 = 0	1 x 2 = 2		
			0 x 1 = 0	1 x 1 = 1	2 x 1 = 2	3 x 1 = 3
			$\boxed{0 \text{ x } 0 = 0}$	1 x 0 = 0		

Questions What entry would you expect in cells adjacent to those shown?

Extend the cell entries upwards and to the right until you can predict the entry in any cell above and to the right of 0 x 0 = 0.

Extend those cell entries downwards. Extend those cell entries to the left.

Now extend the cell entries above and to the left of 0 x 0 = 0 downward; extend the cell entries below and to the right of 0 x 0 = 0 to the left: is there consistency when you extend to the same cell from two directions?

Possible developments Other grid patterns can be established.

IMAGINE

Phenomenon Imagine a point on the plane. Imagine a circle which is a fixed distance from that point. Now imagine two points, and a circle which is equidistant from both points. Now imagine three points, and a circle equidistant from all three points.

Questions In each case, what about the circle can change and still preserve the condition? Where can the centre of the circle get to under the specified condition?

Possible developments Four points? Variations produce classic geometric theorems such as the incircle and the circumcircle of a triangle.

MEAN-MEDIAN-MODE

Phenomenon On a spreadsheet is displayed a histogram of some data points, the mean, median and mode, and possibly the standard deviation. Changing the data points changes the display immediately.

Questions How far apart can the mean, median and mode be within a specified range? How many points can be outside I standard deviation of the mean?

Possible developments Constructing 'data' with specified combinations of mean, median, mode and standard deviation.

FACTORING QUADRATICS

Phenomenon

$x^2 + 5x + 6 = (x + 2)(x + 3)$	$x^2 + 5x - 6 = (x + 6)(x - 1)$
$x^2 - 5x + 6 = (x - 2)(x - 3)$	$x^2 - 5x - 6 = (x - 6)(x + 1)$
$x^2 + 10x + 24 = (x + 6)(x + 4)$	$x^2 + 10x - 24 = (x + 12)(x - 2)$
$x^2 - 10x + 24 = (x - 6)(x - 4)$	$x^2 - 10x - 24 = (x \; -1\,2)(x + 2)$
$x^2 + 17x + 60 = (x + 12)(x + 5)$	$x^2 + 17x - 60 = (x + 20)(x - 3)$
$x^2 - 17x + 60 = (x - 12)(x - 5)$	$x^2 - 17x - 60 = (x - 20)(x + 3)$

Questions Could more be generated?

Possible developments Characterise all quadratics for which all four possibilities of signs of the coefficients factor. What about quadratics with lead coefficient greater than 1? (Minor 1988, Kaczkowski 2001)

There are opportunities to imagine beyond what is seen, to extend and to express generality make this a powerful phenomenon.

Showing some objects, sometimes dynamically, sometimes statically, and asking learners for what is the same and what different, what is changing and what is invariant, is likely to intrigue them sufficiently to want to explain what things happen as they do, or how some pattern might continue. Some phenomena are observed in the material, imagined (including e-screens), symbolic, or social world. Others are created by recognising and exploiting some aspects of the surprise which occasioned the topic in the first place.

PHENOMENAL ONTOLOGY AND AFFECT

What makes me ask questions like these? It is certainly a propensity of mine, but all I do is notice when I am surprised by something, and then ask about the source of that surprise. This is something that everyone can develop. As you pay attention to what surprises you, you also find that you are surprised more and more.

What do I mean by *a phenomenon*? That it is difficult to capture in words is evident from Husserl:

> … the infinity of actual and possible world-experience transforms itself into the infinity of actual and possible 'transcendental experiences', in which, as a first step, the world and the natural experience of it are experienced as 'phenomenon' [Husserl 1970 p153]

I mean some incident or event which is salient and identifiable, and hence which is discerned from the background of experience. By the time something is recognised *as* a phenomenon, there is awareness of generality, of actual or potential similarity and replicability in some form. A phenomenon is created by the observer who discerns, who foregrounds and backgrounds, who stresses and ignores. Other people may not discern in the same way, and even if they discern, they may not be aware of having noticed.

Phenomena, like 'problems' are not intrinsically interesting. It is people in a situation who are interested. Once something discerned is recognised *as* a phenomenon, interest has already been aroused.

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FURTHER TEXT

Material world; mental world, symbolic world; virtual world

Static, dynamic

Must-May-Can't happen

TREATING EXAMPLES

HALF MOON INN

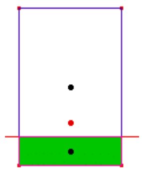
Phenomenon Some years ago I was driving to a workshop I was to give, when I noticed a pub sign similar to the one shown here. The pub was called the half moon inn. Something about the sign caught my attention: something was not right.

Questions Can you ever see a vertical half moon, in the sense that the terminator is a straight line in line with your body as you stand up straight and look at it? If so, when and where would you look; if not, how close can you get?

Possible developments Related problem: can you ever see a horizontal half moon. If so, from where, if not, how close can you get?)

Opportunity to work on imagining and expressing, conjecturing and convincing, as well as to make observations.

TIPSY



Phenomenon One Before you start drinking from a can, the centre of gravity of the can and the liquid is in the middle; as you drink, the centre of gravity drops, following the drop in the surface of the liquid; when you have drunk all the liquid, the centre of gravity of the can is back in the middle where it began.

Questions Somewhere the centre of gravity reaches a minimum height. Where is it?

Possible developments Use of reasoning (no measurements need be made; opportunity to use calculus to find extremal value; use of modelling assumptions.

Phenomenon Two I was playing with a coke can while listening to people talking over lunch, and I accidentally discovered that with only some of the liquid in it, the can balances on its rim. The presence of the liquid gives it an odd motion as it rolls around at this angle.

Questions Somewhere the centre of gravity reaches a minimum height. Where is it?

Possible developments I immediately wondered over what range of volumes of liquid the can would remain stable in this position. What is significant for our purposes here is again, not a model devised to answer this, but rather the sorts of choices made.

Phenomenon Three Work on the drinks can put me in mind of wine racks that I have often seen in shops but never really paused to think about before.

Questions What are the design constraints that make this thing stable for both full and empty bottles?

Possible developments Centres of gravity; combining centres of gravity of different objects to find the centre of gravity of a compound object.

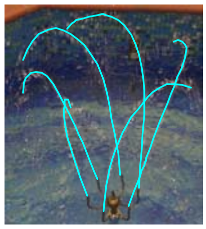
CHECKOUT

In a supermarket, when is it most efficient to start up another checkout?

FOUNTAINS

Phenomenon One Scotch College is an independent school in Melbourne, where I encountered an atrium in the faculty building with a fountain in the centre. The fountain consists of six spouts symmetrically placed at the centre as a rotating wheel. The wheel of spouts rotates, whether solely due to the water, or due to some other mechanism I do not know.

Questions I looked at this a few times, and found myself asking: what am I really seeing? My brain expects parabolae, so that is what I saw … at first. In the second picture I have joined up drops which seem part of the same 'curve'.



Possible developments Start with an ordinary static fountain. It would be possible to use pictures, even video, with the path of single droplets traced, leading to physics modelling of trajectories, then use of calculus or completed square for finding the apex, and the many related problems.

One aim would be to alert learners to the shapes of fountains so that they begin to notice different fountains for themselves, and to be aware of the ubiquitous presence of parabolae.

TIED DOWN

Phenomenon One Consider tying a rope around a rod in a clove hitch. It is a very simple and common knot.

Questions By how much will the rope be shortened?

Possible developments Contrast between empirical approach through collecting data and fitting a curve through it, and a theoretical approach in which an underlying mathematical model is used to make predictions. Newton displayed both of these, the former in his Optics, and the latter in his Principia (Buchdahl 1961).

FUNNEL

Phenomenon	A circle can be folded in half, twice, and then opened to make a conical funnel.
Questions	How much does the funnel hold? How big a cone can be made from a single sheet of A4 paper?
Possible developments	Comparing volumes of cylinders made from a piece of A4 paper. What shape of curve must I cut along to get an elliptical cone from a sheet of paper?

Finding phenomena which can serve as entry to mathematical topics is one source of examples; starting from a topic and seeking relevant phenomena is another. This is most usefully done by looking for the fundamental surprises which underpin the topic: what is it that is not obvious, making this worthy to be considered a topic?